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MATHEMATICIANS AND MUSIC.

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Presidential Address[1] *delivered*[2] *before the Mathematical Association of America, September 6, 1923.*

I.

"Mathematics and Music, the most sharply contrasted fields of intellectual activity which one can discover, and yet bound together, supporting one another as if they would demonstrate the hidden bond which draws together all activities of our mind, and which also in the revelations of artistic genius leads us to surmise unconscious expressions of a mysteriously active intelligence." In such wise wrote one[3] supremely competent to represent both musicians and mathematicians, the author of that monumental work, *On the Sensations of Tone as a physiological basis for the Theory of Music.*

"Bound together?" Yes! in regularity of vibrations, in relations of tones to one another in melodies and harmonies, in tone-color, in rhythm, in the many varieties of musical form, in Fourier's series arising in discussion of vibrating strings and development of arbitrary functions, and in modern discussions of acoustics.

This suggests that the famous affirmation of Leibniz, "Music is a hidden exercise in arithmetic, of a mind unconscious of dealing with numbers,"[4] must

[1] To the address as delivered a number of footnotes, mainly with a few references to the vast literature of the subject, have been added. A fundamental work in this connection is H. L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, and the best edition is the second English edition translated with many additions from the fourth (last) German edition by A. J. Ellis, London, 1885; the third edition was reprinted from the second in 1895, and the fourth from the third in 1912. While some mathematical discussion occurs in this work, the standard treatise on the mathematical theory is Rayleigh, *Theory of Sound*, 2 vols., second ed., London, 1894. Another work of high order is H. v. Helmholtz, *Vorlesungen über die mathematischen Akustik*, 1898, vol. 3 of *Vorlesungen über theoretische Physik*, Leipsic. H. Lamb, *The Dynamical Theory of Sound*, London, 1910, was intended as a stepping stone to the writings of Helmholtz and Raleigh. Between 1898 and 1915, 38 papers by various authors appeared at Leipsic in 8 Hefte of *Beiträge zur Akustik und Musikwissenschaft* herausgegeben von C. Stumpf. For the most part, they are reprints of articles in *Zeitschrift für Psychologie, Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, and *6. Kongress der Gesellschaft für experimentelle Psychologie*. Another very valuable general work, discussing the writings of mathematicians on musical matters, is F. J. Fétis, *Biographie Universelle des Musiciens et Bibliographie générale de la Musique*, 8 vols., second ed., Paris, 1 (1873), 2 (1867), 3–4 (1869), 5–8 (1870). R. Eitner's *Biographisch-bibliographisches Quellen-Lexikon der Musiker der christlichen Zeitrechnung bis zur Mitte des 19. Jahrhunderts*, 10 vols., Leipsic, 1900–1904, is also sometimes useful.

[2] At a joint session of the Mathematical Association of America and of the American Mathematical Society, Vassar College, Poughkeepsie, N. Y.

[3] H. v. Helmholtz, *Vorträge und Reden*, Braunschweig, vol. 1, 1884, p. 82. See also Helmholtz, *Popular Lectures on Scientific Subjects*, London, 1873, p. 62.

[4] "Musica est exercitium arithmeticæ occultum nescientis se numerare animi," which occurs in a letter dated April 17, 1712, and addressed to Goldbach. It is letter 154 in Leibniz, *Epistolæ ad diversos*, vol. 1, Leipsic, 1734, p. 241. The quotation is in the section dealing with the question "Vnde oritur ex musica voluptas?" This is preceded and followed by two other sections on

be far from true if taken literally. But, in a very general conception of art and science, its verity may well be granted; for, in creating as in listening to music, there is no realization possible except by immediate and spontaneous appreciation of a multitude of relations of sound.

Other modes of expression and points of view were suggested by that great enthusiast to whom America owed much, him who called himself [1] "the Mathematical Adam" because of the many mathematical terms he invented; for example, mathematic—to denote the science itself in the same way as we speak of logic, rhetoric or music, while the ordinary form is reserved for the applications of the science. He referred to the cultures of mathematics and music "not merely as having arithmetic for their common parent but as similar in their habits and affections." [2] "May not Music be described," he wrote, "as the Mathematic of Sense, Mathematic as the Music of reason? [3] the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music,—Music the dream, Mathematic the working life,—each to receive its consummation from the other when the human intelligence, elevated to the perfect type, shall shine forth glorified in some future Mozart-Dirichlet, or Beethoven-Gauss —a union already not indistinctly foreshadowed in the genius and labors of a Helmholtz"! [4]

But such intimacies in these cultures are not discoveries and imaginings of a later day. For two thousand years music was regarded as a mathematical science. Even in more recent times the mathematical dictionaries of Ozanam,[5] Savérien,[6] and Hutton,[7] contain long articles on music and considerable space is devoted to the subject in Montucla's revised history,[8]—which brings us to

"Quibus musicis proportionibus homines delectantur?" and "Quando rationes surdæ in musica commode locum inueniunt?"

T. W. Preyer corrupted Leibniz's sentence into "Arithmetica est exercitium musicum occultum nescientis se sonos comparare animi" (compare M. Lecat, *Pensées sur la Science, la Guerre, et sur des Sujets très variés*, Brussels, 1919, p. 438). Preyer's thought in this connection will be apparent on turning to his monograph, "Ueber den Ursprung des Zahlbegriffes aus dem Tonsinn und über das Wesen der Primzahlen," pages 1–36 of *Beiträge zur Psychologie und Physiologie der Sinnesorgane, Hermann von Helmholtz als Festgruss zu seinem siebzigsten Geburtstag.* Gesammelt und herausgegeben von A. König, Hamburg and Leipsic, 1891.

[1] J. Sylvester, in a footnote to "Note on a Proposed Addition to the Vocabulary of Ordinary Arithmetic," *Nature*, vol. 37, p. 152, 1887.

[2] J. Sylvester, British Assoc. for the Adv. of Science, *Report*, 1869, page 7 of Notices and Abstracts.

[3] Compare "Die Mathematik ist die Musik des Verstandes, die Musik die Mathematik des Gefühls" as employed by Josef Petzval, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 12, 1903, p. 327.

[4] This passage occurs in a footnote in the midst of Sylvester's memoir "Algebraical researches containing a disquisition on Newton's rule for the discovery of imaginary roots . . .," *Philosophical Transactions* for 1864, vol. 154, 1865, p. 613.

[5] J. Ozanam, *Dictionaire Mathematique*, Amsterdam, or Paris, 1691.

[6] A. Savérien, *Dictionnaire Universel de Mathematique et de Physique*, Paris, 1753, vol. 2.

[7] C. Hutton, *A Philosophical and Mathematical Dictionary*, London, 1795, vol. 2; new edition, 1815.

[8] J. F. Montucla and J. de La Lande, *Histoire des Mathématiques*, Paris, vols. 1 and 4, 1799, and 1802. Compare D. E. Smith, "The threatened loss of the second edition of Montucla's History of Mathematics" in this MONTHLY, *1921*, 207–208.

the threshold of the nineteenth century. It is, therefore, not surprising that many mathematicians wrote on musical matters. I shall presently consider these at some length. But certain other facts may first be reviewed.

The manner in which music, as an art, has played a part in the lives of some mathematicians is recorded in widely scattered sources. A few instances are as follows.

Maupertuis was a player on the flageolet and German guitar and won applause in the concert room for performance on the former.[1] At different times William Herschel served as violinist, hautboyist, organist, conductor, and composer (one of his symphonies was published) before he gave himself up wholly to astronomy.[2] Jacobi had a thorough appreciation of music.[3] Grassmann was a piano player and composer, some of his three-part arrangements of Pomeranian folk-songs having been published; he was also a good singer and conducted a men's chorus for many years.[4] János Bolyai's gifts as a violinist were exceptional and he is known to have been victorious in 13 consecutive duels where, in accordance with his stipulation, he had been allowed to play a violin solo after every two duels.[5] As a flute player De Morgan excelled.[6] The late G. B. Mathews knew music as thoroughly as most professional musicians; his copies of Gauss and Bach were placed together on the same shelf.[7] It was with good music that Poincaré best liked to occupy his periods of leisure. The famous concerts of chamber music held at the home of Emile Lemoine during half a century exerted a great influence on the musical life of Paris.[8] And in

[1] *Basler Jahrbuch*, 1910, p. 46 in "Maupertuis" (pp. 29–53) by F. Burckhardt; see also "Maupertuis' Lebensende" by the same author in *Basler Jahrbuch*, 1886, pp. 153–159. These very interesting articles contain new material concerning the closing days of Maupertuis at the home of his friend Johann Bernoulli the second. Compare D. E. Smith, "Maupertuis and Frederick the Great" in this MONTHLY, *1921*, 430–432. The first memoir presented to the French Academy by Maupertuis was "Sur la forme des instruments de musique," *Mémoires de l'Academie Royale des Sciences, 1724*, Paris, 1726, pp. 215–226 + 1 plate; *Histoire*, pp. 90–92. This memoir is not contained in *Oeuvres de Mr. Maupertuis* published at Dresden in 1752 and at Lyons in 1756.

[2] Concerning the musical activities of Herschel, see especially *The Scientific Papers of Sir William Herschel*, vol. 1, London, 1912, pp. xiv–xxii; F. J. Fétis, *Biographie*, etc., vol. 4, *loc. cit.*, and A. Noyes, *The Torch-Bearers, Watchers of the Sky*, New York, 1922, p. 231 f. Noyes perpetuates the old error about Herschel "deserting from the army." Herschel was born in 1738 and died in 1822.

[3] S. Hensel, *Die Familie Mendelssohn, 1729–1847*, second ed., Berlin, 1880, vol. 2, pp. 364–365; English edition, London, 1881, vol. 2, p. 324.

[4] Compare F. Engel, *Grassmann's Leben* in *Hermann Grassmann's Gesammelte Mathematische und Physikalische Werke*, vol. 3, part 2, Leipsic, 1911, pp. 250–253, 371–372. Grassmann was born in 1809 and died in 1877.

[5] F. Schmidt, "Lebensgeschichte des ungarischen Mathematikers Johann Bolyai," *Abhandlungen zur Geschichte der Mathematik*, Heft 8, 1898, p. 141. Bolyai was born in 1802 and died in 1860; sketches by G. B. Halsted appeared in this MONTHLY, *1896*, 1–5, and *1898*, 35–38.

[6] Sophia E. De Morgan, *Memoir of Augustus De Morgan*, London, 1882, p. 16. See also A. M. Stirling, *William De Morgan and his Wife*, London, 1922, p. 61. De Morgan was born in 1806 and died in 1871.

[7] *Proceedings of the Royal Society*, London, vol. 101 A, 1922, p. xiv; also in *Proceedings of the London Mathematical Society*, 2 series, vol. 21, 1923, p. l. Mathews was born in 1861 and died in 1922.

[8] L. Augé de Lassus, *La Trompette. Un demi-siècle de Musique de Chambre*, Paris, 1911. See also D. E. Smith, "Emile Michel Hyacinthe Lemoine," in this MONTHLY, *1896*, 29–33. Lemoine was born in 1840 and died in 1912.

America we have only to recall colleagues in the mathematics departments of the Universities of California, Chicago and Iowa, and of Cornell University, who are, to use Shakespeare's phrase, "cunning in music and mathematics."

While Friedrich T. Schubert, the Russian astronomer and mathematician, played the piano, flute, and violin in an equally masterly fashion,[1] his great-grand-daughter Sophie Kovalevsky was devoid of musical talent; but she is said to have expressed her willingness to part with her talent for mathematics could she thereby become able to sing.[2] Abel had no interest in music as such, but only for the mathematical problems it suggested. His close attention to a performer at a piano was once explained by the fact that he sought to find a relation between the number of times that each key was struck by each finger of the player.[3] Lagrange welcomed music at a reception because he could by the fourth measure become oblivious to his surroundings and thus work out mathematical problems; for him the most beautiful musical work was that to which he owed the happiest mathematical inspirations.[4] Dirichlet seemed to be sensible to the charms of music in a similar manner.[5]

Such are a few instances, which could be considerably multiplied, of the relation of mathematicians to the art of music

> "that gentlier on the spirit lies
> Than tir'd eyelids upon tir'd eyes."

They suggest the accuracy of at least a part of the following observations of Möbius in his book on mathematical abilities:[6] "Musical mathematicians are frequent . . . but there are wholly unmusical mathematicians and many more musicians without any mathematical capability." That there are musicians with some mathematical ability will be granted when we recall, not only that Henderson, the prominent New York music critic and the author of many works on musical topics, has written a little book on navigation,[7] but also that the late Sergei Tanaïeff, pupil of Rubenstein and Tchaikovsky, successor of the latter as professor of composition and instrumentation at the Moscow Conservatory of Music, and one of the most prominent of modern Russian composers, found algebraic symbolism and formulæ of fundamental importance in his lectures and work on counterpoint.[8]

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[1] *Allgemeine deutsche Biographie.* Schubert was born in 1758 and died in 1825.

[2] Sophie von Adelung, *Deutsche Rundschau*, vol. 89, 1896, p. 405. Sophie Kovalevsky was born in 1850 and died in 1891.

[3] *Niels Henrik Abel Memorial publié à l'occasion du centenaire de sa naissance*, Christiania, 1902, pp. 57–58. Abel was born in 1802 and died in 1829.

[4] *Œuvres de Lagrange*, Paris, vol. 1, 1867, p. xlviii. Lagrange was born in 1736 and died in 1813.

[5] *G. Lejeune Dirichlets Werke*, vol. 2, Berlin, 1897, p. 343. Dirichlet was born in 1805 and died in 1859.

[6] P. J. Möbius, *Ueber die Anlage zur Mathematik*, zweite vermehrte und veränderte Auflage, Leipsic, 1907, p. 124.

[7] W. J. Henderson, *The Elements of Navigation*, New York, 1895; new and enlarged edition, 1917. He was formerly lieutenant in the first battalion, naval militia of New York.

[8] A. E. Hull, "Music and Mathematics," *The Monthly Musical Record*, London, May 1, 1916, vol. 46, p. 133 f. Tanaïeff was born in 1856 and died in 1915. See also "Russian Music, A Taneyef Souvenir," *The Monthly Musical Record*, vol. 46, pp. 313–314; and *Life and Letters*

A question which has interested more than one group of inquirers is: Can one establish any relationship between mathematical and musical abilities? Within the past year two Jena professors, Haecker and Ziehen, published the results of an elaborate inquiry as to the inheritance of musical abilities in *musical* families.[1] As a by-product of the inquiry they arrived at the result that in only about 2 per cent. of the cases considered was there any appreciable correlation between talent for music and talent for mathematics; they found also that the percentage of males lacking in talent for music but showing a talent for mathematics was comparatively high, about 13 per cent. At the Eugenics Record Office of Cold Spring Harbor, Long Island, there has been collected a considerable body of data upon which a study of the correlation of mathematical and musical abilities could be based. It will be interesting to see if the conclusions of Haecker and Ziehen are here checked, and also if some results are found as to the extent to which musical abilities are present in a group of mathematicians.[2]

II

Turning now to the *theory* of music, it is natural to inquire: What are the relations of mathematics to music? What have mathematicians written about music or its theory? Even on the part of one fully informed and competent, to answer these questions with any degree of completeness would require not one hour only, but many hours. I shall therefore limit myself to brief statements, with references to only a score or so of the better known mathematicians.

In any consideration of the history of music and its relation to mathematics it is important to have in mind the general character of music of different periods. With Helmholtz [3] these may be stated as follows:

(*a*) The Homophonic or Unison Music of the ancients, including the music of the Christian era up to the eleventh century, to which also belongs the existing music of Oriental and Asiatic nations.

(*b*) Polyphonic Music of the middle ages, with several parts, but without regard to independent musical significance of the harmonies, extending from the tenth to the seventeenth century, when it passes into

(*c*) Harmonic or Modern Music, characterized by the independent significance attributed to the harmonies as such.

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of Peter Ilich Tchaikovsky, by M. Tchaikovsky, edited from the Russian by R. Newmarsh, London, 1916, with many references.

[1] "Ueber die Erblichkeit der musikalischen Begabung, nebst allgemeinen methodologischen Bemerkungen uber die psychische Vererbung," *Zeitschrift für Psychologie*, 1922, vol. 88, pp. 265–307; vol. 89, pp. 273–312; vol. 90, pp. 204–306; see particularly pp. 290–298. This was reprinted in book form with the title, *Zur Vererbung und Entwicklung der musikalischen Begabung*, Leipsic, 1922, 3 + 186 pages. Haecker is also the author of *Der Gesang der Vögel, seine anatomischen und biologischen Grundlagen*, Jena, 1900, 6 + 102 pp.

[2] The correlations between arithmetic and singing, space intuition and singing and other subjects studied by a group of 42 boys in the sixth year of a grammar school at Kiel, Germany have been set forth in M. Lobsien, "Korrelationem zwischen den unterrichtlichen Leistungen einer Schülergruppe," *Zeitschrift für experimentelle Pädagogik*, Leipsic, vol. 11, 1910, pp. 146–164

[3] *Loc. cit.*, p. 236.

Our first consideration is therefore to be given to the homophonic music of the *Greeks:* for in music as in mathematics the period of real development began in the sixth century B.C. with Pythagoras. Before his time tones an octave or a fifth apart, above and below, were regarded as consonant and as the basis of ordinary needs in declamation. If the *c* be taken as a point of departure, its fifth is *g*, and its fifth below is *f*. If this last note *f* be raised an octave so as to bring it nearer to the other notes, and if the octave of *c* be also added, the following four notes are obtained: *c, f, g, c*. Tradition affirms that these four notes constituted the range of the lyre of Orpheus. As Blaserna remarks,[1] "Musically speaking it is certainly poor, but the observation is interesting that it contains the most important musical intervals of declamation. In fact, when an interrogation is made, the voice rises a fourth. To emphasize a word, it rises another tone and goes to the fifth. In ending a story, it falls a fifth, etc. Thus it may be understood that Orpheus' lyre, notwithstanding its poverty, was well suited to a sort of musical declamation."

The notable contribution of Pythagoras was his enunciation of the law governing such sounds which are found in all the musical scales known. "He proclaimed the remarkable fact, of which the proof existed in his famous experiments with stretched strings of different lengths, that the ratios of the intervals perceived as consonant could all be expressed by the numbers 1, 2, 3, 4. His method of demonstration was afterward improved and rendered more exact by the invention of the monochord, and his law may now be stated as follows:[2]

"If a string be divided into two parts by a bridge, in such a manner as to give two consonant sounds when struck, the lengths of those parts will be in the ratio of two of the first four positive integers. If the bridge be so placed that two thirds of the string lie to the right and one third to the left, so that the two lengths are in the ratio of 1 : 2, they produce the interval of the octave, the greater length being given to the deeper note. If the bridge be so placed that three fifths of the string lie to the right and two fifths to the left, the ratio of the two lengths is 2 : 3 and the interval produced is the fifth. If the bridge be again shifted to a position which gives four sevenths on the right and three sevenths on the left, the ratio is 3 : 4 and the interval is the fourth." Thus corresponding to the successively higher notes *c, f, g,* and *c* we have the numbers 1, 3/4, 2/3, and 1/2 for the relative lengths of the strings corresponding to the different notes.

The fourth and fifth gave the means of fixing a much smaller interval, called a tone, corresponding to which is the number $8/9 = 2/3 \div 3/4$. Starting with a fundamental *c* and inserting two tones between it and its fourth, two more between its fifth and its octave, the corresponding numbers for the succession *c d e f g a b c* would be 1, 8/9, 64/81, 3/4, 2/3, 16/27, 128/243, 1/2. The numbers corresponding to successive *pairs* of notes would be 8/9, 8/9, 243/256, 8/9, 8/9,

[1] P. Blaserna, *The Theory of Sound in its relation to Music*, London, 1875, p. 117. Compare Helmholtz, *loc. cit.*, p. 255, and D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, p. 113 f.

[2] H. E. Wooldridge, *The Oxford History of Music*, vol. 1, Oxford, 1901, p. 11.

8/9, 243/256, the 243/256 being that number by which it is necessary to multiply into $8/9 \times 8/9$ in order to give 3/4.

Pythagoras looked upon the diatonic scale to which we have just referred in quite a different manner, namely, as derived from a succession of fifths. Thus starting from a prime c we have

$$c\ g\ d\ a\ e\ b.$$

Reducing d an octave, a an octave, e two octaves, and b two octaves, we have the series

$$c\ d\ e\ \ g\ a\ b.$$

To obtain the f missing in this series and to fill up the wide interval between e and g it appears that c as a fifth below the prime was raised an octave. It may be readily verified that we are thus led to the same results as before; for example, d, the second fifth above the prime, is given by $2/3 \times 2/3$; to the d an octave lower corresponds $2 \times 2/3 \times 2/3 = 8/9$.

Pythagoras proposed to find in the order of the universe, where whole numbers and simple ratios prevail, an answer to the question: Why is consonance (the beautiful in sound) determined by the ratio of small whole numbers? The correct numerical ratios existing between the seven tones of the diatonic scale corresponded, according to Pythagoras, to the sun, moon and five planets, and the distances of the celestial bodies from the central fire, etc.

"It was the elaboration of these figments of philosophy, and because the fifth as the central tone of the octave corresponded to the astronomical order in which the Samian sage ranged the sun and planets, that he laid such a deep stress upon the c scale obtained from fifths only." [1]

Pythagoras limited himself to the insertion of seven notes within the octave. But from the primal scale he evolved six others. This was done not by setting up a new succession of fifths on the several notes of the primal scale but by making the second note of his first scale the prime of his second and so for each of five remaining notes. In this way, for example, we get the scale d, e, f, g, a, b, c, d with the corresponding numbers 1, 8/9, 27/32, 3/4, 2/3, 16/27, 9/16, 1/2. To the succession b, c, d, e, f, g, a, b corresponds 1, 243/256, 27/32, 3/4, 729/1024, 81/128, 9/16, 1/2.

It is not apparent in this latter scale that the method of Pythagoras can be said to illustrate the principle that the beautiful in sound must depend upon a succession of notes related to each other and a prime, by the simplest possible ratios.

The most noted of all the musical theorists of antiquity was Aristoxenus of Tarantum, a contemporary and pupil of Aristotle. To him as author have been assigned no less than 453 works but of these none now remain except the Harmonics,[2] portions of a treatise on rhythm, and some fragments recently found in

[1] H. Wylde, *The Evolution of the Beautiful in Sound*, London, 1888. Compare "The music of the spheres," *Harper's Magazine*, vol. 63, 1881, pp. 286–288. But best of all in this connection see chapter 12 of T. L. Heath, *Aristarchus of Samos, the Ancient Copernicus*, Oxford, 1913, with references to still more elaborate discussions.

[2] ΑΡΙΣΤΟΞΕΝΟΥ ΑΡΜΟΝΙΚΑ ΣΤΟΙΧΕΙΑ. *The Harmonics of Aristoxenus, Edited with*

Egypt. According to Macran (page 87), his great service was rendered "firstly, in the accurate determination of the scope of musical science, lest on the one hand it should degenerate into empiricism, or on the other hand lose itself in mathematical physics; and secondly, in the application to all questions and problems of music of a deeper and truer conception of the ultimate nature of music itself."

Of two treatises on music attributed to Euclid,[1] only the Theory of Intervals or Section of the Canon, as it is sometimes called, may be regarded as genuine.[2] It is based on the Pythagorean theory of music, "is mathematical, and clearly and well written, the style and form of the propositions agreeing well with what we find in the Elements."

The way in which the work starts out seems somewhat remarkable when we remember that it was written about three hundred years before Christ. It commences as follows:[3] "If all things were at rest, and nothing moved, there must be perfect silence in the world; in such a state of absolute quiescence nothing could be heard. For motion and percussion must precede sound; so that as the immediate cause of sound is some percussion, the immediate cause of all percussion must be motion. And whereas of vibratory impulses or motions causing a percussion on the ear, some there be returning with a greater quickness which consequently have a greater number of vibrations in a given time, whilst others are repeated slowly and of consequence are fewer in an assigned time, the quick returns and greater number of such impulses produce the higher sounds, whilst the slower which have fewer courses and returns, produce the lower. Hence it follows, that if sounds are too high they may be rendered lower by a diminution of the number of such impulses in a given time, and that sounds which are too low, by adding to the number of their impulses in a given time, may be made as high as we choose. The notes of music may be said then to

translation, notes, introduction, and index of words by H. S. Macran, Oxford, 1902. See also L. Laloy, *Aristoxène de Tarente et la musique de l'Antiquité*, Paris, 1904; C. F. A. Williams, *The Aristoxenian Theory of Musical Rhythm*, Cambridge, 1911; and F. A. Wright, *The Arts in Greece*, London, 1923, pp. 52–55.

[1] The best text with Latin translation, for the musical works attributed to Euclid, is that edited by H. Menge in *Euclidis Opera Omnia* edited by Heiberg and Menge, Leipsic, vol. 8, 1916. There is a critical introduction, "De scriptis musicis," pages xxxvii–liv. These same works, in Greek and Latin, are to be found in David Gregory's edition of Euclid's works (Oxford, 1703, pp. 531–536). A Latin-French edition by P. Herigone appears in his *Cursus Mathematicus*, vol. 5, Paris (1637), 1644, pp. 802–856. There is a French translation by P. Forcadel, *Le livre de la musique d'Euclide*, Paris, 1566; this is also in L. Lucas, *Une révolution dans la musique . . .*, Paris, 1849, and in L. Lucas, *L'Acoustique nouvelle . . .*, Paris, 1854. Another French translation is by C. E. Ruelle, *L'introduction harmonique de Cléonide, La division du canon d'Euclide . . .*, Paris, 1884. An English translation appears in C. Davey, *Letters addressed chiefly to a young gentleman upon subjects of Literature: including a translation of Euclid's section of the Canon; and his treatise on Harmonic; with an explanation of the Greek musical modes according to the Doctrine of Ptolemy*, Bury St. Edmunds, 1787, vol. 2, pp. 264–410. The Theory of Intervals, in Greek, is also given in K. v. Jan, *Musici Scriptores Græci*, Leipsic, 1895–1898, vol. 1, pp. 148–166; there is a "prolegomena" (in Latin), by Jan, pp. 115–147.

[2] Compare Heath, *The Thirteen Books of Euclid's Elements*, Cambridge, 1908, vol. 1, p. 17, and vol. 2, p. 295.

[3] C. Davey, *loc. cit.*, pp. 264–265; the punctuation and slight changes in the wording have been made in the quotation.

consist of parts, inasmuch as they are capable of being rendered precisely and exactly tunable, either by increasing or diminishing the number of the vibratory motions which excite them. But all things which consist of numerical parts when compared together, are subject to the ratios of numbers, so that musical sounds or notes compared together, must consequently be in some numerical ratio to each other."

Nearly two thousand years passed before Galileo went one step further,[1] and proved that the lengths of strings of the same size and tension were in the inverse ratios of the numbers of the vibrations of the tones they produced.[2] It was not for another seventy years that the actual number of vibrations corresponding to a given tone was determined;[3] but we shall return to this a little later.

Euclid's work contains 19 theorems. They are mostly concerned with results which may be obtained by the division of a monochord, or string to be experimented upon, which Euclid calls Proslambanomenos. Let this be named A.[4]

This string A was first divided into four parts; three parts were taken and the perfect fourth established with the ratio 3 : 4; two parts were taken and the sound of the octave established; one part was taken and the sound of the double octave A was given.

The next experiment was to divide the length which produced the fourth of the prime into two equal parts, when the sound, the octave of the fourth, was established.

Proslambanomenos was then divided into two equal parts, and one of these being again divided into three parts, two parts were taken and the octave of the fifth was established.

And so till all the tones in two octaves were determined. By beginning with different letters in the series thus determined, Euclid got the seven Pythagorean scales covering two octaves instead of one. Euclid arrived at these sounds by the division of the monochord instead of by successions of fifths employed by Pythagoras.

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[1] Galileo Galilei, *Discorsi e Dimostrazioni Matematiche*, Leyden, 1638, at the end of the "first day." A new English translation by H. Crew and A. de Salvio appeared at New York in 1914 with the title: *Dialogues concerning Two New Sciences*. The manuscript of the original work was sent to the printer in 1636 and the printing was completed in 1637; but many of the results were given by Galileo in lectures long before.

[2] Credit for this result is given to Galileo with full knowledge of the claims of Mersenne. Compare F. Rosenberger, *Die Geschichte der Physik*, part 1, Braunschweig, 1882, pp. 35–36.

In J. C. Poggendorff, *Histoire de la Physique*, translated by E. Bibart and G. de la Quesnerie, Paris, 1883, the following sentence occurs on page 487: "Deux siècles après *Pythagore*, *Aristote* écrivit sur les sons, et fit preuve d'une connaissance exact des faits, donts il est peut-être redevable aux Pythagoriciens. Il savait, par example, que dans les cordes de tension égale et dans les tuyaux, le nombres des vibrations est en raison inverse des longueurs, et que les sons sont produits par des vibrations qui passent des corps sonores à l'air qui les transmet à notre oreille." I can find no verification of this statement as to Aristotle's knowledge.

[3] Compare Newton, *Philosophiæ Naturalis Principia Mathematica*, London, 1687, book 2, section 8, prop. 50, pp. 369–372; second ed., Cambridge, 1713, pp. 342–344.

[4] Compare "Euclid's mathematical divisions of a string, and resulting series of sounds" in H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, pp. 84–93. See also T. P. Thompson, *Theory and Practice of Just Intonation*, London, 1850, pp. 79–80; fourth ed., 1860, pp. 104–108. The fourth edition has the title *On the Principles and Practice of Just Intonation*.

Two of Euclid's theorems prove that an octave is less than six tones, the ratio of the interval being $(8/9)^6 \div (1/2) = 524288/531441$, or nearly 80 : 81.0915. This same ratio is got from $(2/3)^{12} \div (1/2)^7$. In other words it is the ratio determined by the difference of tones derived by counting 12 fifths and 7 octaves from a fundamental. This interval, between notes theoretically the same, was noted by Pythagoras and is called a Pythagorean comma.[1]

The scales of Pythagoras and Euclid differ in two important respects from our major scales, namely, in the ratios for the intervals of a third and a sixth. In the scale of *c*, the interval of a major third from the tonic is now $4/5 = 64/80$ instead of the Pythagorean $64/81 = (8/9)(8/9)$. This substitution of 4/5, even though not mentioned by Euclid, is not modern, but was already suggested in the late Pythagorean school.[2] The second substitution of 3/5 for the major sixth interval from the tonic naturally followed from this, since it is the octave of the fifth below the third. In this way the ratios of the intervals of the major scale became[3] 1, 8/9, 4/5, 3/4, 2/3, 3/5, 8/15, 1/2, while the intervals between successive pairs of notes became 8/9, 9/10, 15/16, 8/9, 9/10, 8/9, 15/16.

In such a scale if we tune up four perfect fifths on the one hand and two octaves and a major third on the other, we ought to arrive at the same note. The resulting comma here is 80/81 instead of the 80/81.0915 already referred to. It is the distribution of this comma which is ordinarily carried through in our equal-tempered scale. This temperament is said to have been proposed by Aristoxenus.

And last among the Greeks to whom we shall refer is the celebrated mathematician, astronomer and geographer, Claudius Ptolemy, who flourished in the second century of the Christian era. Apart from the Almagest, works on optics and mechanics, a book on stereographic projection, a book in which he tried to show that the possible number of dimensions is limited to three, and other works, Ptolemy wrote a remarkable treatise on music.[4] In it he discusses critically the earlier Pythagorean and Aristoxenean modes and tonalities and presents new developments. But the restrictions made in connection with the music seem to indicate the beginning of a decline.

Some interesting suggestions have been made by Paul Tannery as to the

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[1] Compare Helmholtz, *On Sensations . . .*, *loc. cit.*, p. 432.

[2] D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, p. 123.

[3] If in the series of ratios for the major scale we substitute 5/6 for a minor third, instead of 4/5 as for the major, and 9/16 for the 8/15, we have the succession at which Newton arrived, in an experiment with the prismatic colors of pure light published in his *Optiks*, London, 1704, p. 92. Measuring from an origin to the left to determine the points 1, 8/9, 5/6, 3/4, 2/3, 3/5, 9/16, 1/2, and erecting cross lines he found, as he states, "the said cross lines divided after the manner of musical chord . . . to represent the Chords of that Key, and of a Tone, a third Minor, a fourth, a fifth, a sixth Major, a seventh, and an eighth above the Key: And the intervals . . . will be spaces which the several Colours (red, orange, yellow, green, blue, indiao, violet) take up."

[4] A somewhat defective text of this work, together with a Latin translation by John Wallis, was published at Oxford in 1680. Compare Fétis, *Biographie . . .*, vol. 7. See also *The Oxford History of Music*, vol. 1, by H. E. Wooldridge, Oxford, 1901, pp. 15–22; H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, chapter XI, etc.; and D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, pp. 108–112.

possible rôle of Greek music in the development of pure mathematics.[1] One of these is to the effect that the idea of logarithms may have been suggested by such mathematical relations as the following going back to Pythagoras:

$$\frac{1}{2}=\frac{2}{3}\times\frac{3}{4} \qquad \frac{1}{2}=\left(\frac{3}{4}\right)^2\times\frac{8}{9}$$

being immediately interpreted in music by: The octave is composed of a fifth and a fourth; the octave is composed of two fourths and of a major tone. Thus mathematical multiplication is changed into musical addition.

Another of Tannery's suggestions involves finding solutions of a Diophantine equation in three variables. In the first four notes of the major scale we had the relation

$$\frac{8}{9}\times\frac{9}{10}\times\frac{15}{16}=\frac{3}{4}.$$

Ptolemy derived many scales[2] in which the relations were similar; for example,

$$\frac{7}{8}\times\frac{9}{10}\times\frac{20}{21}=\frac{8}{9}\times\frac{7}{8}\times\frac{27}{28}=\frac{9}{10}\times\frac{10}{11}\times\frac{11}{12}=\frac{3}{4}.$$

In other words the question of the composition of the tetrachord reduces to the following mathematical problem: "Determine all possible ways of decomposing the ratio 3/4 into a product of three ratios of the form $n/(n+1)$." From these results, those were finally selected which seemed practicable after trial with the monochord.

In my brief sketch of the work done by the Greeks, I have not intended to give you any idea of their music, but merely to select a few illustrations of the manner in which their music is connected with mathematics. On the varieties of their scales and their coloring through chromatics (as the name implies) and quarter tones, I have not touched. Nor have I commented on the great beauties of the music even though it was homophonic. Authorities agree with the following summing up of Helmholtz:[3] "Of course where delicacy in any artistic observations made with the senses come into consideration, moderns must look upon the Greeks in general as unsurpassed masters. And in this particular case they had very good reason and abundance of opportunity for cultivating their ears better than ours. From youth upwards we are accustomed to accommodate our ears to the inaccuracies of equal temperament, and the whole of the former variety of tonal modes, with their different expression, has reduced itself to such an easily apprehended difference as that between major and minor. But the varied gradations of expression, which moderns attain by harmony and modulation, had to be effected by the Greeks and other nations that used homo-

[1] P. Tannery, "Du rôle de la musique grecque dans le developpement de la mathématique pure," *Bibliotheca Mathematica*, series 3, vol. 3, 1902, pp. 161–175; also in P. Tannery, *Mémoires Scientifiques*, vol. 3, 1915, pp. 68–89.

[2] Compare "Examen des series d'Archytas" in L. Laloy, *Aristoxène de Tarente et la musique de l'antiquité*, Paris, 1904, pp. 364–365.

[3] *On Sensations . . ., loc. cit.*, p. 266.

phonic music by a more delicate and varied gradation of tonal modes. Can we be surprised, then, if their ear became much more finely cultivated for differences of this kind than it is possible for ours to be?"

The next outstanding figure in our survey is Boetius who flourished in the early part of the sixth century of our Christian era. He was a Roman senator and a philosopher,—"the last of the Romans whom Cato or Tully could have acknowledged for their countryman," as Gibbon expresses it. Not only did Boetius exert great influence in his own time through his summaries of logical and scientific works of the ancients, but for six centuries after his death they were the leading authorities. He wrote works on arithmetic, geometry and music. While the first printed edition of the arithmetic appeared at Venice in 1488, all three united seem to have first been published in 1492. Details of the mathematical works have been given by Cantor.[1] His extensive treatise on music[2] is a valuable repertory of the knowledge of the ancients in this art. It was long used as a text at the Universities of Oxford and Cambridge.[3] Boetius sets forth the details of the accomplishments of the Pythagoreans and the teachings of such writers as Aristoxenus which were opposed to those of Pythagoras. He also surveys the Ptolemaic musical scheme in connection with those of Pythagoras and Aristoxenus. Since the doctrine of Boetius was mainly Pythagorean, this was the system which prevailed for centuries later.

As the Roman absorbed the Greek, so the Christians accepted the Roman organization of learning. In the medieval curriculum the scope of this learning on the secular side was comprised within the seven liberal arts[4] and philosophy. The seven liberal arts, divided into the *Trivium* (grammar, dialectic, rhetoric), and the more advanced *Quadrivium* (geometry, arithmetic, music and astronomy), were an inheritance from a period at least as early as the second century before Christ; indeed the Quadrivium division of mathematical studies is Pythagorean.[5] Some explanation of the nature of the subjects of the Trivium is necessary in order to make their scope clear; but we are only concerned with the mathematical sciences of the Quadrivium in which, early in the middle ages, the course in geometry was more a course in geography and surveying than in the subject matter of Euclid's *Elements* which later became a text. The study of music consisted mainly in becoming acquainted with the mathematics of the subject, and with the mystic properties of its numbers,—much as taught by the Pythagoreans. As a liberal art it concerned itself neither with singing (apart from its rules), nor with playing on an instrument. Astronomy with its practical applications to the calendar and sun dial was the most popular of the Quadrivium subjects but there was probably more of astrology in it than astronomy as we now understand the term.

[1] *Vorlesungen über Geschichte der Mathematik*, vol. 1, third ed., Leipsic, 1907. See also D. E. Smith, *Rara Arithmetica*, Boston, 1908.

[2] See Fétis, *Biographie . . .*, vol. 1, *loc. cit.*, and Eitner, *Quellen-Lexikon*, vol. 2, *loc. cit.* *Arithmetica et Musica* of Boetius, ed. by Friedlein, Leipsic, 1867, is the standard edition.

[3] See article "Boetius" in *Encyclopædia Britannica*, eleventh edition.

[4] Compare P. Abelson, *The Seven Liberal Arts*, New York, 1906.

[5] T. L. Heath, *A History of Greek Mathematics*, Oxford, 1921, vol. 1, pp. 11–12.

Some four hundred years after the time of Boetius, the polyphonic period in the development of music had its inception in the composition of certain two-part song-forms. During the six hundred years which followed, that is, till towards the close of the sixteenth century, polyphonic music adorned with canon, fugue, and counterpoint was developed to a notable degree.

Of mathematicians who flourished in this period I shall refer to only one, Girolamo Cardano.[1] Among those of the sixteenth century achieving a reputation in mathematics and medicine none was better known than he, whose greatest mathematical work, *Ars Magna* (1545), contains the first solution of the general cubic equation in print.

Cardano was an ardent lover of music and while living in Milan his house was constantly filled with men and boys of somewhat sinister reputation but capable of joining with him in part-singing so popular in the polyphonic period. During the last twenty-five years of his life he spent considerable time in writing a work on music,[2] which was in many respects original and must have been welcomed by all musical students as a valuable contribution to the literature of the subject. This work begins by laying down at length the general rules and principles of the art, and then goes on to treat of ancient music in all its forms; of music as Cardano knew and enjoyed it; of the system of counterpoint and composition, and of the construction of musical instruments.[3]

An interesting glimpse of Cardano's personality may be gleaned in another place from his listing of the joys of home and children. Incidentally he suggests: "Let the young child . . . be shut out from the sight or hearing of all ill. When he is about seven years old let him be taught elements of geometry to cultivate his memory and imagination. With syllogisms cultivate his reason. Let him be taught music, and especially to play upon stringed instruments; let him be instructed in arithmetic and painting, so that he may acquire taste for them, but not be led to immerse himself in such pursuits. He should be taught also a good hand-writing, astrology, and when he is older, Greek and Latin."

In the early part of the *third* period in the development of music, namely, the period of Harmonic or Modern Music, we have the first opera and the first oratorio, and, as I have already said, the discovery by Galileo that the simple ratios of the lengths of strings existed also for the pitch numbers of the tones they produced, an observation later generalized by Newton. By the time of Rameau, the most eminent French composer and writer on the theory of music in the eighteenth century, the harmonics or upper partial tones of the human voice had been recognized and made the basis for more satisfying harmonic development. A string, for example, vibrates not only as a whole but also, at

[1] 1501–1576.

[2] G. Cardano, *Opera Omnia*, Leyden, 1663, vol. 4. See also fragment no. 6 in vol. 10. Volume 4, no. 10 is *Opus novum de proportionibus numerorum, motuum, ponderum, sonorum aliarumque rerum mensurandarum . . .* first published at Basle in 1570.

[3] W. G. Waters, *Jerome Cardan, a biographical Study*, London, 1898, p. 256; see also pp. 163, 235. In the more extensive biography by H. Morley, *The Life of Girolamo Cardano of Milan, Physician*, 2 vols., London, 1854, there are references to music on the following pages: vol. 1, pp. 41, 45, 202, 295; vol. 2, pp. 19, 43, 53.

the same time, in each of its aliquot parts 1/2, 1/3, 1/4, 1/5, 1/6, and so on. Thus the first upper partial tone is the upper octave of the prime tone, the second is the fifth of this octave, the third upper partial is the second higher octave, the fourth is the major third of this second higher octave, the fifth is the fifth of the second higher octave, making six times as many vibrations as the prime in the same time; and so on, each successive upper partial tone being fainter than the preceding. It may be shown that beginning with the twenty-fourth upper partial all the notes of a major scale may be obtained from the dominant, that is, the fifth.[1] The dominant and not the tonic is thus the root, of the whole scale. In the bugle, trumpet, French horn, and other instruments only the fundamental tone of the instrument and some of its harmonics can be sounded. On a horn about four feet long the notes are c, c', g', c'', e'', and g'', —the primes denoting tones in higher octaves.

Not all upper partials need exist in connection with a fundamental musical tone. Certain tuning forks have no upper partials.[2] In 1800, the noted physicist, Thomas Young, who first furnished the key to decipher Egyptian hieroglyphics, was also the first to show that "when a string is plucked or struck, or, as we may add 'bowed' at any point in its length which is the node of any of its so-called harmonics, those simple vibrational forms of the string which have a node in that point are not contained in the compound vibrational form. Hence if we attack at its middle point, all the simple vibrations due to the even numbered partials, each of which has a node at that point, will be absent. This gives the sound of the string a peculiarly hollow or nasal twang."[3] Because of this law piano makers eliminate[4] certain undesirable upper partials by striking the middle strings of their instruments at a point 1/7 to 1/9 of their lengths from their extremities. So too in making other instruments it is possible to eliminate, or reinforce, certain partials.

But we have got ahead of our story. Returning to the beginning of the Harmonic period let us consider the musical writings which were issued in the seventeenth century by such mathematicians as Kepler, Wallis, Mersenne, Desargues, Descartes and Christian Huygens.

Pythagorean ideas on the ratios of numbers and of proportions applied to the constitution of the universe seem to have been the point of departure of Kepler in his famous work *Harmonices Mundi* published in 1619.[5] It is in the fifth book of this work that one first finds the third fundamental law of modern astronomy, "The squares of the periodic times of the several planets are proportional to the cubes of their mean distances from the sun," demonstration of which furnished Newton with the basis for his theory of gravitation. The third book of the work is especially devoted to music and it may be characterized

[1] The scale is made from the following partials: 24, 27, 30, 32, 36, 40, 45, 48.

[2] Helmholtz, *On Sensations . . ., loc. cit.*, pp. 54, 528.

[3] Helmholtz, *On Sensations . . ., loc. cit.*, p. 52.

[4] Helmholtz, *On Sensations . . ., loc. cit.*, p. 77; but compare pp. 545–546.

[5] *Joannis Kepleri astronomi opera omnia*, ed. C. Frisch, vol. 5, Frankfurt, 1864. Compare Fétis, *Biographie . . ., loc. cit.*, vol. 5. Kepler was born in 1571 and died in 1630.

as mainly a work on the philosophy of music. The fifth book to which I have referred is somewhat allied to the third, since in it the author endeavored to establish curious analogies between the harmonic proportions of music and astronomy.

Markedly contrasted to Kepler in abilities and habits of thought was John Wallis, the notably able Savilian professor at Oxford University, where a brilliant mathematical school was developed under his direction. He is well known as mathematician and cryptographer,[1] but few have observed his extensive writings on musical matters [2] filling more than 500 folio pages in the third volume of his collected works. The first of these is a Greek and Latin edition of Ptolemy's Harmony, and Porphyry's third century commentary [3] on the same, with an extensive appendix by Wallis on ancient and modern music. Then comes the only published text, with Latin translation, of a musical work by Manuel Bryenne, a fourteenth century Greek, four manuscripts of whose work are to be found at the Bodleian. Among other writings of Wallis on acoustics and music may be mentioned four memoirs published in the *Philosophical Transactions*,[4] and bearing the following titles: "On the trembling of consonant strings," "On the division of the monochord, or section of the musical canon," "On the imperfections of an organ," and "On the strange effects of music in former times."

The Franciscan friar Marin Mersenne, Wallis's senior by nearly 30 years, is known to the general run of mathematicians through the numbers with which his name is associated and which arise in discussion of perfect numbers. He was widely acquainted with French and foreign contemporary mathematicians and actively corresponded with them. His work in physics [5] dealt chiefly with questions in acoustics. He determined ratios of the vibration numbers of strings varying in thickness and tension, results included in those of Brook Taylor derived mathematically about 70 years later. I have not been able to verify the statement [6] that Mersenne noticed, but attached no importance to the observation, that a vibrating string gave forth not only the fundamental tone but also higher sounds. We have already remarked that Rameau made much of the fact in the following century. Mersenne wrote half a dozen works on harmony and musical instruments [7] but his most notable one is *L'Harmonie Universelle*, a great work of 1500 pages with an immense quantity of engraved plates and musical examples. This was published in 1636–7. It is really a combination of several treatises, for example, On the Nature of Sounds and

[1] Compare D. E. Smith, "John Wallis as a Cryptographer," *Bulletin of the American Mathematical Society*, vol. 24, pp. 82–96, 1917. Wallis was born in 1616 and died in 1703.

[2] Compare Fétis, *Biographie . . .*, vol. 8, *loc. cit.* Also H. Mendel, *Musikalisches Conversations-Lexikon*, vol. 11, 1878.

[3] Jan believes that this was probably mostly compiled by Pappus or some other competent mathematician; see K. v. Jan, *Musici Græci Scriptores Græci*, Leipsic, 1895, p. 116.

[4] 1677–1698.

[5] F. Rosenberger, *Die Geschichte der Physik*, part 1, Braunschweig, 1882, pp. 93–95, etc.; also J. C. Poggendorff, *Histoire de la Physique*, Paris, 1883, pp. 488–489.

[6] J. K. Fischer, *Geschichte der Physik*, vol. 1, Göttingen, 1801, pp. 468, 470.

[7] Compare Fétis, *Biographie . . .*, vol. 6, *loc. cit.*; compare Eitner, *Quellen-Lexikon . . .*, vol. 6, *loc. cit.*

Movements of All Sorts of Bodies, On Voice and Songs, and On Instruments. There is also a treatise on mechanics, by Roberval, which no one but a Mersenne could regard as appropriately placed in his work on harmony. While no sections of the work are of transcendent merit, one finds a great amount of information, especially regarding Frenchmen, which is no longer to be found elsewhere. It is only here, for example, that we learn that the geometer Desargues was the author of a method of singing.

Among Mersenne's friends was one, some eight years his junior, René Descartes. That he was interested in music [1] is attested by the fact that a score of his published letters treat of motions of vibrating strings and various musical topics. Moreover in 1618, when 22 years of age, he wrote a *Compendium Musicæ*, but this was first published as a little tract of 58 pages [2] in 1650, the year of his death. The material is arranged under about a dozen headings such as: the object of music is the sound; number and time that one should observe in the sounds; concerning the diversity of sounds; consonances; the octave; the fifth; the fourth; the second, minor third, and sixth; the degrees or tones of music; dissonances; and the manner of composing—in connection with which five principles are laid down in an interesting manner.

A copy of the manuscript of the *Compendium* found its way to one afterwards to become a particular friend of Descartes.[3] This was Constantin Huygens, a many-sided genius possibly best known as a poet and a musician; he was a competent performer on several instruments and author of several musical works. His second son was Christian the great Dutch mathematician, mechanician, astronomer and physicist. Two publications dealing with musical matters were written by Christian Huygens. The first of these is a brief sketch of 1691, entitled "Novus Cyclus Harmonicus," and occupying only 8 quarto pages.[4] In them he suggests another solution of the problem of how suitably to arrive at a tempered scale. If we divide the octave into twelve equal parts or degrees, we have a cycle in which a fifth of 7 and a major third of 4 degrees approximates to Pythagorean intonation. A cycle of 53, with a fifth of 31 and a major third of 18, had also been proposed, and led to similar results. The new harmonic cycle of Huygens contained 31 degrees, with a fifth of 23 and a major third of 10, and closely imitates mean tone temperament.[5] He refers to the writings of

[1] Compare Fétis, *Biographie . . .*, vol. 3, *loc. cit.*; and Eitner, *Quellen-Lexikon . . .*, vol. 3, *loc. cit.* Descartes was born in 1596 and died in 1650.

[2] See also *Œuvres de Descartes* publiées par Charles Adam et Paul Tannery, Paris, vol. 10, 1908, pp. 79–150. An English Translation was published at London in 1653. There were four other French editions.

[3] See many references in *Œuvres de Descartes*, vol. 12, 1910, *Vie & Œuvres de Descartes* by C. Adam. See also in this MONTHLY, *1921*, 167, where in the course of an article by D. E. Smith on "Descartes's appreciation of Huygens the elder" a letter from Descartes dated May 23, 1632, contains the following clause: "I do not know how to respond to the courtesy of Monsieur Huygens, except that I cherish the honor of his acquaintance as one of the greatest pieces of fortune that has come to me."

[4] In *L'Histoire des Ouvrages des Sçavans*, Rotterdam, October, 1691, p. 78; also C. Huygens, *Opera Varia*, Amsterdam, 1724, vol. 3, pp. 747–754. Christian Huygens was born in 1629 and died in 1695.

[5] Compare Helmholtz, *On Sensations . . .*, *loc. cit.*, p. 436. See also *Proc. Roy. Soc. of London*, vol. 13, 1864, p. 412.

Mersenne, and of Zarlino, "one of the most learned and enlightened music theorists of the sixteenth century." I have already drawn attention to the natural way in which logarithms enter into the discussion of musical intervals. So far as I have been able to determine, this little publication of Huygens is the first to illustrate this fact.

The second work of Huygens containing musical material was finished for the press just before his death. Three years later it appeared simultaneously in Latin and English and is an exceedingly entertaining work. It is entitled *The Celestial Worlds discover'd: or Conjectures concerning the Inhabitants Plants and Productions of the Worlds in the Planets.*[1] In order adequately to present an idea of a section on mathematics and music I shall quote somewhat extensively.

The author surmises that if the surfaces of Jupiter and Saturn are divided like ours into sea and land it is reasonable to suppose that the inhabitants must know of the art of navigation. He then infers that they must have the "Mechanical Arts and Astronomy, without which Navigation can no more subsist, than they can without Geometry." Huygens then continues (page 84): "But Geometry stands in no need of being prov'd after this manner. Nor doth it want assistance from other Arts which depend upon it, but we may have a nearer and shorter assurance of their not being without it in those Earths. For that Science is of such singular Worth and Dignity, so peculiarly imploys the Understanding, and gives it such a full Comprehension, and infallible certainty of Truth, as no other Knowledge can pretend to: it is moreover of such a Nature, that its Principles and Foundations must be so immutably the same in all Times and Places, that we cannot without Injustice pretend to monopolize it and rob the rest of the Universe of such an incomparable Study. Nay Nature itself invites us to be Geometricians: it presents us with Geometrical Figures, with Circles and Squares, with Triangles, Polygons, and Spheres, and proposes them as it were to our Consideration and Study which abstracting from its usefulness is most delightful and ravishing. Who can read *Euclid* or *Apollonius*, about the Circle, without Admiration? Or *Archimedes* of the Surface of the Sphere, and Quadrature of the Parabola without Amazement? Or consider the late ingenious Discoveries of the Moderns, with Boldness and Unconcernedness? And all these Truths are as naked and open, and depend upon the same plain Principles and Axioms in *Jupiter* and *Saturn* as here, which makes it not improbable that there are in the Planets some who partake with us in these delightful and pleasant studies." Then a little later the author continues (page 86): "It's the same with Musick as with Geometry, it's everywhere immutably the same, and always will be so. For all Harmony consists in Concord, and Concord is all the World over fix'd according to the same invariable Measure and Proportion so that in all Nations the Difference and Distance of Notes is the same, whether they be in a continued gradual Progression, or the Voice skips over one to the next. Nay, very credible Authors report, that there's a sort of Bird in *America*, that can plainly sing in

[1] London, 1698; "second edition corrected and enlarged" in 1722; "new edition corrected," Glasgow, 1762. The Latin edition published at The Hague is entitled *Κοσμοθεωρός sive de terres cœlestibus, earumque ornatu, conjecturae*; second edition, 1699.

order six musical Notes: Whence it follows, that the Laws of Musick are unchangeably fix'd by Nature, and therefore the same Reason holds for their Musick, as we e'en now shewed for their Geometry."

Discussing the probability of other planets' being inhabited and of the inhabitants' possible interest in music and invention of musical instruments, he continues (page 88): "What if they should excell us in the Theory and practick part of Musick, and outdo us in consorts of vocal and instrumental Musick, so artificially compos'd, that they shew their skill by the Mixtures of Discords and Concords and of this last sort 'tis very likely the 5th and 3d are in use with them.

"This is a very bold Assertion, but it may be true for aught we know, and the Inhabitants of the Planets may possibly have a greater insight into the Theory of Musick than has yet been discover'd among us. For if you ask any of our Musicians why two or more perfect Fifths cannot be used regularly in Composition; some say 'tis to avoid that Sweetness and Lushiousness which arises from the repetition of this pleasing Chord. Others say, this must be avoided for the sake of that Variety of Chords that are requisite to make a good Composition; and these Reasons are brought by Descartes [1] and others. But an Inhabitant of *Jupiter* or *Venus* will perhaps give you a better Reason for this, *viz.* because when you pass from one perfect Fifth to another, there is such a Change made as immediately alters your Key, you are got into a new key before the Ear is prepared for it, and the more perfect Chords you use of the same kind in Consecution, by so much the more you offend the Ear by these abrupt Changes."

It may interest harmony students of our day to learn that the prohibition of consecutive fifths [2] was not something recently invented for their undoing, but was a matter of fundamental importance adequately explained over two hundred years ago. And this is not the only passage of interest for such students in the last work of Christian Huygens.

I have already referred to Thomas Young's memoir of 1800 and his explanations of varied qualities of tone through agitation of a string at different points. The mere fact of such differences of quality had been already noted by Huygens [3] in connection with harpsichords—those precursors of pianos in the sixteenth, seventeenth, and eighteenth centuries.

III.

In the eighteenth century when calculus had become a tool, there was a notable series of theoretical discussions of vibrating strings. But before considering these I wish to draw special attention to the first English scientific treatment of harmony, a work of high order, by Robert Smith.[4] It was entitled *Harmonics or the Philosophy of Musical Sounds* and was first published [5] in 1749.

[1] This was simply "*Cartes*" in the original.

[2] Compare "Modern music and 'fifths,'" *Monthly Musical Record*, vol. 46, 1916, pp. 43–44.

[3] Helmholtz, *On Sensations . . ., loc. cit.*, p. 77.

[4] Born 1689, died 1768. See *Dictionary of National Biography*, Oxford. Eitner, *Quellen-Lexikon . . ., loc. cit.*, vol. 9, confuses this Robert Smith with an earlier musician and composer; see *Musical Antiquary*, Oxford, vol. 2, pp. 171–173, 1911.

[5] "Second edition much improved and augmented," London, 1759, 20 + 293 pp. + 28 plates.

The theory of intervals and various systems of temperament are discussed in a manner very attractive even for a reader in the present day. Smith held the Plumian chair at Cambridge, the one of which A. S. Eddington is the present incumbent, and his work on harmonics contained the substance of lectures he had delivered for many years. It was he who was the author of the notable work on *Optics* which has been translated into several languages. He was also the founder of the well-known Smith's Prizes "annually awarded to those candidates who present the essays of greatest merit on any subject in mathematics or natural philosophy." [1]

First in the series of theoretical discussions to which I have referred are those of Brook Taylor, who, according to his biographer,[2] "possessed considerable ability as a musician and an artist." His discussions appeared in the *Philosophical Transactions* [3] for 1713 and 1715 and in his book *Methodus Incrementa Directa et Inversa*,[4] the first treatise dealing with finite differences, and the one which contains the celebrated theorem regarding expansions, now connected with Taylor's name. He solved the following problem which he believed to be entirely new: "To find the number of vibrations that a string will make in a certain time having given its length, its weight, and the weight that stretches it." In discussing the *form* of the vibrating string, his suppositions regarding initial conditions, including that it vibrated only as a whole, led to a differential equation whose integral gave a sine curve. Thus started a discussion which was to culminate a century later in the work of Fourier.

I have already referred to the discovery of upper partial tones by Rameau and how he made this the basis of a system of harmony; his first work on this subject was published in 1726, but the first mathematician who seemed to take account of the fact was Daniel Bernoulli in a memoir of 1741–43 though not published [5] till 1751. About this time D'Alembert's thorough acquaintance with Rameau's theories was shown by his publication in 1752 of a volume entitled "Elements of theoretical and practical music according to the principles of Monsieur Rameau, clarified, developed, and simplified." Of this work six French editions [6] and one in German were published.[7] Helmholtz remarks [8] that D'Alembert's book "is an extremely clear and masterly performance, such

A *Postscript* (12 pp. + 1 plate) was published in 1762. A German edition was published at Berlin in 1771.

[1] *Cambridge University Calendar*, 1921–22.

[2] E. I. Carlyle in *Dictionary of National Biography*, Oxford. Taylor was born in 1685 and died in 1731.

[3] Vol. 28, London, 1714, "De motu tensi," pp. 26–32; vol. 29, 1715, "An account of a book entitled Methodus Incrementorum by the author," pp. 339–350.

[4] London, 1715; other title pages are dated 1717.

[5] D. Bernoulli, *Comment. acad. sc. Petrop.*, vol. 13 (1741/3), 1751, p. 173, § 8. Bernoulli was born in 1700 and died in 1782.

[6] *Eléments de musique théorique et pratique, suivant les principes de M. Rameau, éclaircis, développés et simplifiés*, Paris, 1752; second edition, 1759; third and fourth editions, Lyons, 1762, 1766, 1772, and 1779. D'Alembert was born in 1717 and died in 1783. Compare Eitner, *Quellen-Lexikon* . . ., vol. 1, *loc. cit.*

[7] The German edition by F. W. Marpurg was published at Leipsic in 1757.

[8] *On Sensations* . . ., *loc. cit.*, p. 232.

as was to be expected from a sharp and exact thinker, who was at the same time one of the greatest physicists and mathematicians of his time. Rameau and D'Alembert lay down two facts as the foundation of their system. The first is that every resonant body audibly produces at the same time as the prime its twelfth and next higher third as upper partials. The second is that the resemblance between any tone and its octave is generally apparent. The first fact is used to show that the major chord is the most natural of all chords, and the second to establish the possibility of lowering the fifth and the third by one or two octaves without altering the nature of the chord, and hence to obtain the major triad in all its different inversions and positions."

D'Alembert wrote also a long essay on the liberty of music [1] and articles of musical interest in the great *Encyclopédie Methodique*. But from a mathematical point of view, his memoir of 1747 dealing [2] with Taylor's problem of the vibrating string, and taking account of matters previously overlooked, is very notable. He was led to the differential equation (with a, a constant, equal to unity)

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2},$$

where the origin of coördinates was at the end of the chord whose length is l, the axis of x in the direction of the chord, and y the displacement at any time t. Of this equation he found the solution

$$y = f(at + x) - f(at - x),$$

where f represents any function such that $f(z) = f(z + 2l)$. He then found certain equations for determining the functions satisfying this relation of periodicity.

Euler immediately raised the question of the generality of the solution and set forth *his* interpretation. D'Alembert had supposed the initial form of the string to be given by a single analytical expression, while Euler regarded it as lying along any arbitrary continuous curve, different parts of which might be given by different analytical expressions. Lagrange [3] joined in the discussion, to which Daniel Bernoulli contributed chiefly from physical rather than mathematical considerations. He started with Taylor's particular solution and found, in effect, that the function for determining the position of the string after starting from rest could n aturally be expressed in a form later called a Fourier series.[4] Thus

[1] "*De la liberté de la musique*" in his *Mélanges de Littérature et de Philosophie*, Amsterdam, 1767–1773, and reprinted in d'Alembert, *Œuvres Philosophiques, historiques et littéraires*, Paris, 1805, vol. 3, pp. 335–409. The first sentence gives a clue as to the meaning of the title. It is: "Il y a chez toutes les nations, deux choses qu'on doit respecter, la *religion* et le *gouvernement;* en France on y en ajoute une troisième, la musique du pays."

[2] "Recherches sur la courbe que forme une corde tendue mis en vibration," *Memoires de l'Acad. Royale des Sciences et belles Lettres, 1747*, 1749, pp. 214–219; "Suite des Recherches sur . . .," pp. 220–249.

[3] Euler (1707–1783); Lagrange (1736–1813).

[4] The series is not stated explicitly by Bernoulli in the memoir, "Sur le mêlange de plusieurs espaces de vibrations simples isochrones, qui peuvent coexister dans un même système de corps," *Histoire de l'Acad. Royale des Science, 1753*, Berlin, 1755; but it can be put together from different places, no. 17, p. 160, and no. 23, p. 165.

were such series first introduced into mathematical physics. Bernoulli remarked that since his solution was perfectly general it should include those of Euler and D'Alembert. In this way mathematicians were led to consideration of the famous problem of expanding an arbitrary function as a trigonometric series. No mathematician would admit even the possibility of its solution till this was thoroughly demonstrated, in connection with certain problems in the flow of heat, by Fourier who gives due credit to the suggestiveness of the work of those in the previous century to whom I have referred. Fourier's results were contained in a memoir crowned by the French Academy in 1812 but not printed till more than a decade later. It is sometimes asserted that the first mathematical proof of Fourier's results, with the limits of arbitrariness of the function carefully stated, was given by Dirichlet in his classic memoirs of 1829 and 1837. So far as the limits of arbitrariness are concerned this is correct; but that Fourier rigorously established his expansion of an arbitrary function seems to admit of no denial or qualification.[1]

One of Euler's most notable papers connected with the history of Fourier's series did not appear in print till 1793, ten years after his death. Thus for eighty years, from Taylor to Euler and Lagrange, mathematicians were occupied with the problem of the vibrating string[2] and allied problems including the vibration of a column of air and of an elastic rod. Then thirty years of silence and the great advance by Fourier.

I have indicated only a few bald facts since details in this regard are readily available elsewhere.[3]

Although more than twenty years Fourier's senior, Gaspard Monge,[4] so well known as an expounder of the applications of analysis to geometry,[5] and of descriptive geometry, was associated with him in more than one undertaking. They were professors at the École Polytechnique in Paris, which Monge was largely instrumental in founding. They both accompanied Napoleon to Egypt where Monge was the first president of the Institute of Egypt and Fourier its secretary. Monge was a passionate devotee of music and made a journey to Italy in order to procure copies of all of the musical works in the chapel of St. Mark's, Venice. He was also an ardent republican and, according to Arago,[6] an enthusiast for the "Marseillaise" which he sang every day at the top of his

[1] Compare Darboux, in *Œuvres de Fourier*, vol. 1, Paris, 1888, p. 512.

[2] D'Alembert's own account of this is interesting. See his article "Cordes, (*vibrations des*)" in his *Encyclopédie Methodique, Mathématiques*, vol. 1, Paris, 1784.

[3] I have found Burkhardt's monumental report to be the most complete and most reliable source of information. It is entitled "Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik," *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 10, Heft 2, 1901–1908.

[4] Compare Fétis, *Biographie . . .*, vol. 6, *loc. cit.* See also D. E. Smith, "Monge and the American colonies," in this MONTHLY, *1921*, 166. Monge was borne in 1746 and died in 1818.

[5] When 25 years of age Monge published his "Recherche des équations des surfaces d'apres leurs mode de génération" in the memoirs of the academy of Turin which had been founded through Lagrange's influence. On reading this paper Lagrange exclaimed, "Avec son application de l'analyse à la représentation des surfaces, ce diable d'homme sera immortel." (D. F. J. Arago, *Œuvres Complètes*, vol. 2, Paris, 1854, pp. 447–448.)

[6] Arago, *Œuvres*, vol. 2, p. 516.

voice before seating himself at the table. He, too, occupied himself with the problem of the vibrating string [1] and constructed a model of a surface, certain parallel sections of which give the form of the curve of the vibrating string at any time under conditions which Monge states. This model which was made in 1794 is still preserved in the École Polytechnique.

And finally in connection with great mathematicians of the eighteenth century, the extent of Euler's contributions to the theory of vibrating bodies, acoustics, and music, may be indicated somewhat further.[2] About 30 of his published memoirs, and a treatise, *Tentamen novæ theoriæ musicæ*,[3] not to speak of letters in his Letters to a German Princess, deal with such subjects. They appeared during about 60 years [4] from the first, a dissertation on sound, published in 1727, when he was 20 years old. Among the topics of memoirs not already referred to are: On the sound of bells, Conjectures as to the reason of some dissonances generally accepted in music, The true character of modern music, and On the vibratory motion of drums. It is in this last mentioned memoir of 1766 that the general so-called Bessel's functions of integral order first occur.

Euler's treatise on music was first published in 1739, but we learn from a letter Euler wrote to Daniel Bernoulli in May, 1731, that he had already almost completed the manuscript of the work. This letter describing the ideals of the work in some detail, as well as Bernoulli's reply in the following August, are readily accessible.[5] I shall therefore make but brief extracts from the early parts of the letters. Euler explains, "My main purpose was that I should study music as a part of mathematics and deduce in an orderly manner, from correct principles, everything which can make a fitting together and mingling of tones pleasing. In the whole discussion I have necessarily had a metaphysical basis, wherein the cause is contained why a piece of music can give one pleasure and the basis for it is to be located, and why a thing to us pleasing is to another displeasing." To this Bernoulli replied, "I cannot readily divine wherein that principle should exist, however metaphysical it may be, whereby the reason could be given why one could take pleasure in a piece of music, and why a thing pleasant for us, may for another be unpleasant. One has indeed a general idea of harmony that it is charming if it is well arranged and the consonances are well managed; but, as it is well known, dissonances in music also have their use since by means of them the charm of the immediately following consonances is brought out the better, according to the common saying *opposita juxta se posita magis elucescunt* [opposites placed together shine brighter]; also in the art of painting, shadows must be relieved by light."

[1] Monge, "Construction de l'équation des cordes vibrantes," *Journal de l'Ecole Polytechnique*, cahier 15, tome 8, Paris, 1809, pp. 118–145.

[2] Compare Fétis, *Biographie . . .*, vol. 3, *loc. cit.*

[3] First published at St. Petersburg in 1739; there were French editions in 1839 and 1865.

[4] These are all listed on pages 319–321, 332–333, of G. Enestrom's remarkable *Verzeichnis der Schriften Leonard Eulers*, Leipsic, 1910, 1913.

[5] *Bibliotheca Mathematica*, third series, vol. 4, 1903, pp. 383–388. Bernoulli's letter was published earlier in *Journal für Mathematik*, vol. 23, 1842, pp. 199–200, and in *Correspondance mathématique et physique . . .*, ed. by Fuss, St. Petersburg, 1843, vol. 2, pp. 8–11.

Euler's treatise does not seem to have met with unqualified favor. Brewster reports Fuss to have said [1] "it had no great success as it contained too much geometry for musicians, and too much music for geometers." Helmholtz gives a good deal of space [2] to setting forth the psychological considerations which Euler explains had influenced him to found his relations of consonances to whole numbers.

But here we must leave this "myriad-minded" eighteenth century genius.

And now there is time for but the briefest references to mathematics and music during the past one hundred years—the century in which niceties of mathematical calculation were surely contributory to the improvement of such instruments as the flute and organ, to the wonders of phonograph-record manufacture, of broadcasted concerts, and of sound-wave photography—the century in which Helmholtz and Rayleigh lived and worked.

Helmholtz's epoch-making work, *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik*, first appeared in 1862. The great feature of this work is the formulation and proof of the laws by which the ear hears musical sounds from one or more distinct sources; how the theory of combined musical sounds is reduced to the theory of combined simple sounds. The starting point of these discoveries was the fact, recognized by Rameau just two hundred years ago, that upper partials were associated with fundamental tones. From these laws we learn the *nature* of consonance and dissonance, knowledge so necessary for building up a system of harmony; we learn the principles which determined those degrees of musical sound selected by various nations at various times; we understand the reasons for the simple ratios of the lengths of strings producing consonant tones and the limitation of the numbers of these ratios; and we appreciate the value of temperaments for different instruments.[3]

In his *Tonempfindung* Helmholtz relegated to appendices the purely mathematical discussions. For example, the third appendix is "On the motion of plucked strings;" the fifth is "On the vibrational forms of pianoforte strings"; the sixth is an "Analysis of the motion of violin strings"; and the seventh is "On the theory of pipes." [4] He goes into such matters more extensively in the volume of his lectures on the mathematical principles of acoustics.[5]

Such subjects are also treated in masterly fashion by Rayleigh in his *Theory of Sound* [6] and in his papers. Among other works will be mentioned only the mathematical elements of music, as presented some twenty-five years earlier by Airy,[7] senior wrangler and astronomer.

[1] *Letters of Euler on Different Subjects in Natural Philosophy to a German Princess*, ed. by D. Brewster, third ed., vol. 1, Edinburgh, 1823, p. xxv.

[2] Helmholtz, *On Sensations . . ., loc. cit.*, pp. 229–231. Compare H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, pp. 171–172.

[3] Compare Helmholtz, *On Sensations . . .*, first English edition, 1875, p. vi.

[4] Helmholtz, *On Sensations . . ., loc. cit.*, III, pp. 374–477; V, pp. 380–384; VI, 384–387; VII, pp. 388–396.

[5] Helmholtz, *Vorlesungen . . ., loc. cit.;* see, for example, discussion regarding the violin on pp. 121–139.

[6] *Loc. cit.*

[7] G. B. Airy, *On Sound and Atmospheric Vibrations with the Mathematical Elements of Music*, London and Cambridge, 1868; second edition, 1871.

In such works, in the comparatively recent notable paper in this country by Harvey Davis,[1] on vibrations of a rubbed string, and, of course, in other mathematical treatments of similar material, Fourier series must enter in a fundamental manner. With specified conditions the series and its coefficients for a given tone or combination of tones may be determined. Or, if we have a graph of the vibrations corresponding to such tones, the series may also be calculated, various terms in the series corresponding to simple elements compounded in the tone or tones.

During the past twenty years photography has contributed in a remarkable manner to the analysis of musical sounds. In England, from 1905 to 1912, E. H. Barton and his associates published [2] a series of papers illustrated by photographs of vibration curves particularly as issuing from the violin strings, bridge, and belly.

In India, five years ago, R. C. V. Raman published an extensive bulletin "On the mechanical theory of the vibration of bowed strings and of musical instruments of the violin family, with experimental verification of the results." [3] It is illustrated by reproductions of many photographs, those of the wolf-notes, so well known to stringed-instrument players, having especial interest. The more recent publications of S. Garten and F. Kleinknecht contain a discussion of tones produced by the voice.[4] And with us the work that D. C. Miller, of the Case School of Applied Science, has done in this connection is known to many, not only through his volume on *The Science of Musical Sounds*,[5] but also through his remarkably interesting public lectures where his extraordinary instrument called the phonodeik, which photographically records sound waves, may also be used for projecting traces of the waves, as generated, on the screen of a lecture platform.

For the mathematician a great advantage of a photograph is that he can, after much labor, from it calculate the corresponding Fourier series. But in the laboratory, work of this kind is often saved by the employment of a machine called the harmonic analyzer.[6] The first instrument of this kind was made by Lord Kelvin in 1878; two were put forth by Henrici in 1894, and among others is that of Michelson and Stratton, constructed in 1898. By means of a Henrici machine, when the stylus of the instrument is moved along the curve of the

[1] H. N. Davis, "The Longitudinal Vibrations of a Rubbed String," *Proceedings of the American Academy of Arts and Sciences*, vol. 41, 1906, pp. 691–727 + 3 plates.

[2] *Philosophical Magazine*, 1905–1907, 1909. 1910, 1912.

[3] *Indian Association for the Cultivation of Science, Bulletin no. 15*, Calcutta, 1918, part 1, 158 pp.

[4] "Beiträge zur Vokallehre," I–III, *Abhandlungen der mathematischen-physikalischen Klasse der Sächsischen Akademie der Wissenschaften*, vol. 38, 1921–22; the sub-title for part III is: "Die automatische Analyse der gesungenen Vokale."

[5] New York, 1916.

[6] Compare G. A. Carse and J. Urquhart, "Harmonic Analysis" in *Modern Instruments and Methods of Calculation*, edited by E. M. Horsburgh, London, 1914, pp. 220–247. H. de Morin, *Les Appareils d'Intégration*, Paris, 1913, pp. 147–190. W. Dyck, *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, Munich, 1892, pp. 212–222, and 1893, pp. 34–36.

photograph the numerical values of the coefficients in the corresponding Fourier series may be read off. In 1910 Miller reconstructed a Henrici analyzer so as to care for thirty components with precision.[1] That is, a tone made up of 30 simple tones can be analyzed and the coefficients of the corresponding number of terms in the Fourier series written down. Regarding this kind of work I must not pause to do more than suggest that it has applications of high importance for tone generation and for perfecting musical instruments.

In concluding references to activities of the past one hundred years, I should, however, take time to recall that when, in these latest days, there arose a question as to the manner in which our present musical notation for equal temperament scales could best be simplified, it was a former president of this Association who brought forward a scheme[2] so beautifully simple that further advance in this regard cannot be imagined.

Speculation as to music of the future furnishes tempting themes for discussion. I shall merely mention some of these in conclusion.

The possibilities of melody and harmony in the trinity of musical fundamentals have, within the limits of our hampering scale systems, been largely explored. But what is to be the future of the almost untried vast *rhythmic* possibilities so intimately bound up with mathematical relations? Practically all of our music is modulo 2, 3, 4, 6, 8, 9, 12; but why not have modulo 5, 7, 10, 11, 13, for example, or combinations of these moduli in the same measures?

Again, is it not within the realms of possibility that some day the inadequacies of the present vehicle of musical expression may lead us to revive some of the ideals of Greek music during the golden period of Aristoxenus?

And yet again, when we recall the many results in connection with musical tones found empirically by makers of musical instruments but for which no satisfactory explanations have been furnished by the mathematician or physicist, may we not conclude that when such explanations are forthcoming, a new era shall have dawned in the evolution of musical instruments?

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[1] D. C. Miller, *loc. cit.*, p. 100.

[2] E. V. Huntington, "A simplified musical notation," *The Scientific Monthly*, vol. 11, 1920, pp. 276–283. The following footnote occurs on p. 282: "By the addition of an occasional single letter (Ellis's 'duodenal'), the new notation can even be made to indicate the note required for 'just intonation' with complete accuracy. A discussion of this phase of the subject is reserved for another occasion." In *Science*, 1921, T. P. Hall endorses the scheme (January 28, pp. 91–92) and R. P. Baker raises several pertinent considerations (March 11, pp. 235–236).

THE GEOMETRY OF RIEMANN AND EINSTEIN.

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PART II

7. Some Metrical Relations. The equations (49), as we have seen, are thus the parametric equations of an R-straight in terms of the parameter s. To the parameter value s' will correspond a point z'. For z to coincide with z' it is necessary that $z_i' = z_i$ or

$$a_i \cos\frac{s}{R} + b_i \sin\frac{s}{R} = a_i \cos\frac{s'}{R} + b_i \sin\frac{s'}{R}; \qquad i = 1, \cdots, 4,$$

or

$$a_i\left\{\cos\frac{s}{R} - \cos\frac{s'}{R}\right\} = b_i\left\{\sin\frac{s'}{R} - \sin\frac{s}{R}\right\},$$

or

$$-a_i \sin\frac{s+s'}{2R}\sin\frac{s-s'}{2R} = b_i \cos\frac{s'+s}{2R}\sin\frac{s'-s}{2R}. \tag{54}$$

Now,

$$\sum a_i b_i = 0.$$

Therefore,

$$\sum a_i b_i \cos\frac{s+s'}{2R}\sin\frac{s-s'}{2R} = 0,$$

or, using (54),

$$\sum a_i^2 \sin\frac{s+s'}{2R}\sin\frac{s-s'}{2R} = 0.$$

Therefore,

$$\sin\frac{s+s'}{2R}\sin\frac{s-s'}{2R} = 0.$$

Thus either $s + s' = 2R\pi n$, or $s - s' = 2R\pi n$, n an integer.

The first of these in (54) leads to $\sin s/R = 0$ which is obviously impossible; we must therefore adopt the second. Hence as s increases from $s = 0$, z' never coincides with z until $s = 2\pi R$. Hence the

THEOREM: *The length of all R-straights is $2\pi R$ in R-measure.*

As in 2-way space we may define a *restricted R-geometry* as follows:

1°. Diametrically opposite points on the fundamental sphere F are to be regarded as one and the same point.

2°. Points outside the fundamental sphere are to be regarded as non-existent or imaginary.

We may denote this geometry by R^*. In this R^*-geometry

Two points determine an R^*-straight.

Two R^*-straights cut once and only once.

The R^*-straight is closed and its length is πR.

The truth of the last statement is seen thus: Let the R^*-straight meet F in A, A'. If A has the coördinates $(a_1, a_2, a_3, 0)$, A' has the coördinates $(-a_1, -a_2, -a_3, 0)$. In (49) let us begin to measure s at A. The point z reaches A' when $s = \pi R$ and not before. As $A = A'$ in R^* this is the length of a straight in R^*.

Let us multiply the equations (49) by a_i and add; we get

$$\Sigma a_i z_i = \cos\frac{s}{R}\Sigma a_i^2 + \sin\frac{s}{R}\Sigma a_i b_i,$$

or using, (50), (51),

$$\cos\frac{s}{R} = \frac{a_1 z_1 + \cdots + a_4 z_4}{R^2}; \qquad s = \text{dist } (a, z). \tag{55}$$

This formula which gives the distance between two arbitrary points a, z is fundamental; it is of course measured along the R-straight joining a and z.

We have seen that

$$u_1 z_1 + \cdots + u_4 z_4 = 0 \tag{56}$$

defines an R-plane. We may call $u_1, \cdots, u_4$ the *plane coördinates* of this plane since when these are given, this plane is determined. As the plane whose coordinates are ku_1, ku_2, ku_3, ku_4 is the same as (56), we may suppose that the u's satisfy the relation:

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = R^2, \tag{57}$$

i.e., the z-coördinates of a point and the coördinates of a plane satisfy the same type of relation (24), (56). This fact is important.

Let us find the angle in R-measure between the plane (56) and

$$v_1 z_1 + \cdots + v_4 z_4 = 0, \tag{58}$$

where the plane coördinates v_i satisfy the relation

$$v_1^2 + \cdots + v_4^2 = R^2. \tag{59}$$

If we pass to the x-coördinates, the two equations (56), (58) become as we have already seen

$$u_4(x_1^2 + x_2^2 + x_3^2) - 4R(u_1 x_1 + u_2 x_2 + u_3 x_3) - 4u_4 R^2 = 0,$$
$$v_4(x_1^2 + x_2^2 + x_3^2) - 4R(v_1 x_1 + v_2 x_2 + v_3 x_3) - 4v_4 R^2 = 0.$$

The E-angle θ between these spheres is given [1] by

$$\cos\theta = \frac{u_1 v_1 + \cdots + u_4 v_4}{R^2}, \tag{60}$$

[1] *Cf.* Snyder and Sisam, *Analytic Geometry of Space*, p. 56.

using (57), (59). As angles in R-measure have the same value as in E-measure, (60) gives the angle in R-measure between the R-planes u, v. The reader should note the similarity between the formula (55) for the distance between two points and the formula (60) for the angle between two planes. On the right side of (55) we have the coördinates of the points; on the right side of (60) the coordinates of the two planes.

From (60) we see that when $\theta = \pi/2$,

$$u_1v_1 + \cdots + u_4v_4 = 0, \tag{61}$$

and conversely.

The distance s between the two points $a = (a_1, \cdots, a_4)$ and $-a = (-a_1, \cdots, -a_4)$ is given by (55), *viz.*,

$$\cos\frac{s}{R} = -\frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{R^2} = -1.$$

Therefore, $s = \pi R$. Thus one of the points a or $-a$ falls within F, the fundamental sphere. For this reason we may regard $(a_1, \cdots, a_4)$ or $(-a_1, \cdots, -a_4)$ as defining the same point in R^*.

8. Poles and Polars. Two points a, z whose distance apart is $s = \pi R/2$ are said to be *conjugate*. Setting this value of s in (55), we get

$$a_1z_1 + \cdots + a_4z_4 = 0. \tag{62}$$

Thus the locus of all points z conjugate to a given point a is an R-plane (62). We say (62) is the polar plane of a and call a the pole of (62). The coördinates $a_1, \cdots, a_4$ of the point a satisfy the relation (24), that is, $a_1^2 + \cdots + a_4^2 = R^2$. The point $-a$ whose coördinates are $-a_1, \cdots, -a_4$ also has (62) as polar plane. Thus $-a$ is also a pole of (62), and a given plane has two poles in R and one pole in R^*. To avoid confusion, we will suppose the following to refer to R^*; the extension to R is obvious. From the above we have the

THEOREM 1. *The polar plane of the point (a_1, a_2, a_3, a_4) has the plane coordinates $a_1, \cdots, a_4$ and the pole of the plane whose coördinates are $a_1, \cdots, a_4$ is the point $a = (a_1, \cdots, a_4)$.*

Let δ be the distance between the two points a, b. Then by (55)

$$\cos\frac{\delta}{R} = \frac{a_1b_1 + \cdots + a_4b_4}{R^2}. \tag{63}$$

The polar planes of a, b are

$$a_1z_1 + \cdots + a_4z_4 = 0,$$
$$b_1z_1 + \cdots + b_4z_4 = 0,$$

and by (60), if θ is the angle between these planes,

$$\cos\theta = \frac{a_1b_1 + \cdots + a_4b_4}{R^2}.$$

Comparing this with (63) gives

$$\cos\frac{\delta}{R} = \cos\theta, \qquad \text{or} \qquad \theta = \frac{\delta}{R}. \tag{64}$$

Hence the

THEOREM 2. *The angle between two planes is $1/R$ the distance between their poles.*

Let us find the distance δ from the point $P = (a_1, \cdots, a_4)$ to the plane

$$u_1z_1 + \cdots + u_4z_4 = 0.$$

The coördinates of the pole Q of this plane are $(u_1, \cdots, u_4)$. Let the R-straight joining Q and P cut this plane in the point U. Then $\overline{QU} = \pi R/2$, since Q, U are conjugate. From $\overline{QP} + \overline{PU} = \overline{QU}$ we have, if we call $\overline{QP} = d$, $d + \delta = \pi R/2$, since by hypothesis $\overline{PU} = \delta$. Then by (55)

$$\cos\frac{d}{R} = \frac{u_1a_1 + \cdots + u_4a_4}{R^2} = \cos\frac{\pi R/2 - \delta}{R} = \sin\frac{\delta}{R}.$$

Thus δ is given by

$$\sin\frac{\delta}{R} = \frac{a_1u_1 + \cdots + a_4u_4}{R^2}. \tag{65}$$

Let α be the polar plane of A; if B is a point of α, the polar plane β of B passes through A. For B lying in α is conjugate to A; as β contains all points conjugate to B it must contain A.

Let the above planes α, β cut [1] in the line l; let C be a point of l and let γ be the polar of C. Then γ passes through the join m of A, B. For C lying on α is conjugate to A; also since C lies on β, it is conjugate to B. But γ containing all points conjugate to C must contain A, B and hence their join m.

We call l, m *reciprocal polars.* The foregoing gives the

THEOREM 3. *When a point describes an R-straight l, its polar plane rotates about the reciprocal polar m of l.* Any point of l is conjugate to any point of m.

We have also the

THEOREM 4. *Any plane through l is perpendicular to any plane through m.*

For their poles L, M lie respectively on m and l and dist. $(L, M) = \delta = \pi R/2$, as L, M are conjugate. This value of δ in (64) gives $\theta = \pi/2$.

Let ABD be the polar plane of C cutting l in D; we show easily that the polar plane of D is the plane ABC. These four points A, B, C, D and their opposite faces form a tetrahedron called a *polar tetrahedron.* The polar plane of each vertex is the opposite face, opposite edges are reciprocal polars, and two faces meeting on an edge are perpendicular.

Let A_1, A_2, A_3, A_4 be a polar tetrahedron, T, whose vertex A_4 is the origin O and whose vertices A_1, A_2, A_3 lie respectively on the x_1, x_2, x_3-axes. The length of the edge OA_i, $i = 1, 2, 3$, in R-measure is $\rho = \pi R/2$; hence by (9) its length in E-measure is $r = 2R \tan \pi/4 = 2R$. Thus the polar plane of $O = A_4$ is in E-geometry the fundamental sphere F. Obviously the z-coördinates of the vertex A_k are all zero except the kth for which $z_k = R$.

[1] The reader is requested to make for himself the figure of a tetrahedron whose vertices are A, B, C, D.

Let now $u_1z_1 + \cdots + u_4z_4 = 0$ be an arbitrary plane; if η_k is the distance of the vertex A_k from this plane, we have by (65)

$$\sin\frac{\eta_k}{R} = \frac{u_kR}{R^2} = \frac{u_k}{R}; \qquad k = 1, 2, 3, 4,$$

since by the foregoing all the terms on the right of (65) except the kth are 0. Thus

$$u_k = R\sin\frac{\eta_k}{R}. \tag{66}$$

Thus the four coördinates of a plane are simple functions of the distance of the four vertices of the above tetrahedron from this plane. There is an analogous interpretation of the z-coördinates of a point. For the plane coördinates of the face α_k opposite the vertex A_k are all 0 by (66) except the kth for which $u_k = R$. Thus if ζ_k is the distance of an arbitrary point of space $z = (z_1, \cdots, z_4)$ to the face α_k, we have by (65)

$$\sin\frac{\zeta_k}{R} = \frac{z_kR}{R^2} \qquad \text{or} \qquad z_k = R\sin\frac{\zeta_k}{R}; \qquad k = 1, \cdots, 4. \tag{67}$$

Thus the four coördinates $z_1, \cdots, z_4$ of a point are simple functions of the distance from this point to the four faces of the above tetrahedron.

Let B_1, B_2, B_3, B_4 be the vertices of another polar tetrahedron T'; let the equation of the face β_k opposite B_k be

$$b_{k1}x_1 + b_{k2}z_2 + b_{k3}z_3 + b_{k4}z_4 = 0; \qquad k = 1, \cdots, 4,$$

where as usual the plane coördinates satisfy the relation

$$\sum_j b_{kj}^2 = R^2; \qquad j = 1, \cdots, 4.$$

Since the faces of T' are mutually perpendicular, we have also

$$\sum_j b_{kj}b_{ij} = 0 \qquad \text{if} \qquad i \neq k. \tag{68}$$

Let ζ_k' be the distance of the point $P(z_1, \cdots, z_4)$ from the face β_k. Then by (65)

$$\sin\frac{\zeta_k'}{R} = \frac{b_{k1}z_1 + \cdots + b_{k4}z_4}{R^2}.$$

Let us set analogous to (67)

$$y_k = R\sin\frac{\zeta_k'}{R}; \tag{69}$$

we may call $y_1, \cdots, y_4$ the coördinates of P relative to the tetrahedron T'. Comparing the last two equations gives

$$y_k = a_{k1}z_1 + \cdots + a_{k4}z_k; \qquad k = 1, \cdots, 4, \tag{70}$$

where we have set $b_{kj} = Ra_{kj}$. Thus when we pass from the original tetrahedron

T to the tetrahedron T', the new coördinates y of a point P are related to the old coördinates z by the homogeneous linear relation (70). This transformation may be represented by the accompanying table A which we read as in ordinary analytic geometry.

	z_1	z_2	z_3	z_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	
y_2	a_{21}	a_{22}	a_{23}	a_{24}	(A)
y_3	a_{31}	a_{32}	a_{33}	a_{34}	
y_4	a_{41}	a_{42}	a_{43}	a_{44}	

The orthogonal relations (67), (68) give

$$\begin{aligned} a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} + a_{i4}a_{j4} &= 0, \quad i \neq j, \\ &= 1, \quad i = j. \end{aligned} \tag{71}$$

Such bilinear expressions of the elements of two rows of the table A are called *scalar products*. We have thus: The scalar product of any two rows of A is 1 or 0 according as the two rows are the same or different. It is easily shown[1] that the same holds for two columns of A. The determinant a of the coefficients of (A) is ± 1. We shall take the $a = +1$.

9. Motion, Displacements. One of the most important notions in E-geometry is that of rigid bodies and their displacement. Rigid bodies may be moved or displaced from one position to another without altering any of their dimensions. Thus if $d\sigma$ is the distance between two adjacent points before displacement and $d\sigma'$ their distance afterwards, $d\sigma' = d\sigma$ for all such pairs. The question at once arises can bodies be displaced in R-geometry so that $ds' = ds$. The answer is "yes." For let us look at the transformation defined by (70) or by the adjoined table (A) in another way. We may in fact regard (70) as a transformation which converts every point of space z into the point $y = (y_1, \cdots, y_4)$ relative to the original tetrahedron T which we regard as fixed. Since a linear relation

$$u_1z_1 + \cdots + u_4z_4 = 0$$

goes over into a linear relation, say

$$v_1y_1 + \cdots + v_4y_4 = 0,$$

after this transformation, we see A transforms R-planes into R-planes and hence R-straights into R-straights. For this reason we call A a collineation.

To show that $ds' = ds$ after (A), we have, differentiating (70),

$$dy_k = \sum_j a_{kj}dz_j.$$

Therefore,

$$dy_k^2 = \sum_j a_{kj}dz_j \sum_i a_{ki}dz_i = \sum_{ij} a_{kj}a_{ki}dz_jdz_i; \qquad i, j = 1, \cdots, 4.$$

[1] See Snyder and Sisam, *loc. cit.*, p. 40.

Thus

$$ds'^2 = \sum_k dy_k^2 = \sum_k \sum_{ij} a_{kj}a_{ki}dz_j dz_i = \sum_{ij} dz_i dz_j \cdot \sum_k a_{ki}a_{kj}.$$

Now the last sum on the right is the scalar product of the ith and jth columns of A, hence this sum vanishes unless $i = j$, in which case it is 1. Thus

$$ds'^2 = \sum_i dz_i^2 = ds^2.$$

We have thus proved the fundamental

THEOREM: *A linear transformation* (70) *subject to the orthogonal conditions* (71) *defines a displacement, that is,* $ds' = ds$.

In particular, we see that it is possible to make a given polar tetrahedron coincide with another given polar tetrahedron.

An immediate application of the free mobility of our figures in R-space enables us to establish the

THEOREM: *The geometry on any R-plane is the same as that on an E-sphere of radius R.*

For the given R-plane α may be made to coincide with the x_1, x_2-plane without altering any dimensions in R-measure and the geometry on this plane we have considered in § 2.

Let us consider the special transformation U defined by the adjoined table. Its determinant $= 1$ and the scalar products of rows or columns are 0 or 1 as required by (71). Hence U is a displacement. Let us see how it moves the points of space.

	z_1	z_2	z_3	z_4
y_1	$\cos\theta$	$\sin\theta$	0	0
y_2	$-\sin\theta$	$\cos\theta$	0	0
y_3	0	0	1	0
y_4	0	0	0	1

(U)

First, we note that each point P on the x_3-axis is left unaltered by U. For by (30) the z-coördinates of P are:

$$z_1 = z_2 = 0, \qquad z_3 = R\sin\rho/R, \qquad z_4 = R\cos\rho/R. \tag{72}$$

After U, P has the same coördinates, hence P is unmoved.

Let us next consider the polar tetrahedron whose vertex A_4 is the origin O, while the other vertices A_1, A_2, A_3 lie on the x_1, x_2, x_3-axes, respectively. The z-coördinates of a point Q on the edge A_1A_2 are by (30)

$$z_1 = R\cos\alpha, \qquad z_2 = R\cos\alpha_2 = R\sin\alpha, \qquad z_3 = z_4 = 0, \tag{73}$$

where we have set $\alpha = \alpha_1$ to save the subscript. On applying U, the point Q has the coördinates:

$$y_1 = z_1\cos\theta + z_2\sin\theta = R(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = R\cos(\alpha - \theta),$$
$$y_2 = -z_1\sin\theta + z_2\cos\theta = R\sin(\alpha - \theta); \qquad y_3 = z_3, \qquad y_4 = z_4.$$

Thus the point Q has been moved through the angle $-\theta$ along the edge A_1A_2, and the plane A_3OQ has rotated about the x_3-axis through this angle.

Let L be an arbitrary point in space; let the R-plane ω through L perpendicular to the x_3-axis cut this axis in M. U leaves ω unchanged as a whole while the R-straight ML is turned through the angle $-\theta$, leaving the length $\overline{LM}$ unaltered. We see U is a rotation about the x_3-axis in the Euclidean sense.

We wish now to consider the analogous displacement V defined by the adjoined table.

	z_1	z_2	z_3	z_4
y_1	1	0	0	0
y_2	0	1	0	0
y_3	0	0	$\cos\theta$	$\sin\theta$
y_4	0	0	$-\sin\theta$	$\cos\theta$

(V)

On applying V to the point Q lying on the edge A_1A_2, whose coördinates are given by (73), we see Q is unmoved. On the other hand, V moves the point P defined by (72) lying on the x_3-axis to the point P' whose coördinates are

$$y_1 = 0, \qquad y_2 = 0,$$
$$y_3 = R\sin\rho/R\cos\theta + R\cos\rho/R\sin\theta,$$
$$y_4 = -R\sin\rho/R\sin\theta + R\cos\rho/R\sin\theta.$$

Thus V leaves the x_3-axis unmoved as a whole. To find the distance δ that P has moved along this axis, we have by (55)

$$\cos\frac{\delta}{R} = \frac{y_1z_1 + \cdots + y_4z_4}{R^2} = \cos\theta.$$

Therefore, $\delta = R\theta$. As this is independent of the position of P on the x_3-axis, we have the

THEOREM: *The displacement V rotates all R-planes through the edge A_1A_2 through the angle θ; the poles of these planes P move through the distance $\delta = R\theta$ along the x_3-axis.*

We see the displacement V is a rotation about the edge A_1A_2 analogous to the displacement U relative to the edge A_3A_4.

Let us now see the effect of applying first U and then V to the points of space. U converts $z_1, \cdots, z_4$ into

$$y_1 = z_1\cos\theta + z_2\sin\theta, \qquad y_2 = -z_1\sin\theta + z_2\cos\theta, \qquad y_3 = z_3, \qquad y_4 = z_4,$$

while V converts the point $y_1, \cdots, y_4$ into the point z' say, whose coördinates are:

$$\begin{aligned} z_1' &= z_1\cos\theta + z_2\sin\theta, & z_2' &= -z_1\sin\theta + z_2\cos\theta, \\ z_3' &= z_3\cos\theta + z_4\sin\theta, & z_4' &= -z_3\sin\theta + z_4\cos\theta, \end{aligned} \tag{74}$$

which we can represent by the subjoined table.

	z_1	z_2	z_3	z_4
z_1'	$\cos\theta$	$\sin\theta$	0	0
z_2'	$-\sin\theta$	$\cos\theta$	0	0
z_3'	0	0	$\cos\theta$	$\sin\theta$
z_4'	0	0	$-\sin\theta$	$\cos\theta$

(W)

The determinant of W is 1 and its rows satisfy the condition of orthogonality (71); W is thus a displacement.

Since W is the result of the application of U and then V, we see that now all points on both the edges A_1A_2 and A_3A_4 are moved along these edges through the distance $\delta = R\theta$. This is true not only for these points, but for all points of space. For let $a = (a_1, \cdots, a_4)$ be an arbitrary point. On applying W it goes over into the point z whose coördinates are:

$$\begin{aligned} z_1 &= a_1\cos\theta + a_2\sin\theta, & z_2 &= -a_1\sin\theta + a_2\cos\theta, \\ z_3 &= a_3\cos\theta + a_4\sin\theta, & z_4 &= -a_3\sin\theta + a_4\cos\theta. \end{aligned} \tag{75}$$

If we regard θ as variable, these equations are the parameter equations of the path that the point a describes. The distance δ between a and z is of course measured along the R-straight joining a and z; and as yet we do not know that this straight is the same as the path curve (75). However by (55) we have

$$\cos\frac{\delta}{R} = \frac{a_1z_1 + \cdots + a_4z_4}{R^2}.$$

On substituting the values of $z_1, \cdots, z_4$ given in (75) this gives

$$\cos\frac{\delta}{R} = \cos\theta.$$

Therefore, $\delta = R\theta$, which was to be proved.

Let ds be an element of arc on the path curve (75), we wish to show that

$$ds = Rd\theta \qquad \text{or} \qquad \frac{ds}{d\theta} = R. \tag{76}$$

In fact let z, z' be the two points on (75) corresponding to θ and $\theta + d\theta$. Then

$$\cos\frac{ds}{R} = \frac{z_1z_1' + \cdots + z_4z_4'}{R^2} = \cos d\theta.$$

Therefore, $ds/R = d\theta$, which is (76).

To show that the path curves (75) are R-straights we employ the equations (48) which define straights. Changing the variable from s to θ in (48) by using (76) we have

$$\frac{d^2z_i}{d\theta^2} + z_i = 0, \qquad i = 1, \cdots, 4. \tag{77}$$

The curve (75) will be a R-straight if $z_1, \cdots, z_4$ as there given satisfy (77). That this is so is seen at once on differentiating twice the equation (75). We have thus the

THEOREM: *The displacement W causes all the points of space to describe R-straights of length $R\theta$. All R-planes through the edge A_3A_4 are rotated through an angle $-\theta$ while planes through the edge A_1A_2 rotate through the angle θ. It is thus a double rotation.*

10. Clifford Parallels. We have seen that all straights in an R-plane meet, there are no parallels in this Euclidean sense. From another standpoint there are; in fact, one property of E-parallels is that their distance apart is constant. Such straights exist in R-geometry and are called *Clifford parallels* after their discoverer.

Let P be an arbitrary point of space; from this point drop[1] an R-perpendicular on the x_3-axis meeting this in the point L. We now apply the transformation W. In allowing θ to vary from $\theta = 0$ to $\theta = \theta$, the point P describes an R-straight, moving to P' say, such that $\overline{PP'} = R\theta$. Meanwhile L moves along the x_3-axis

[1] The reader is requested to draw the figure.

to L' such that $\overline{LL'} = R\theta$ also. The R-straight PL goes over into the R-straight $P'L'$ also perpendicular to the x_3-axis and $\overline{P'L'} = \overline{PL}$ in R-measure. Thus the path curve l which P describes under the displacement W is an R-straight such that each point of l is at a constant R-distance from the x_3-axis which is also an R-straight. These two lines are thus parallel in the sense of Clifford.

There is another Clifford parallel through the given point P which we obtain as follows. In the rotation V let us replace θ by $-\theta$, we get a rotation V_1 about the edge A_1A_2 in the opposite direction to V. If we combine this with U we get the double rotation W_1 defined in the adjoined table. This is entirely analogous to W. The equations of the path curve through the point a are analogous to (75):

	z_1	z_2	z_3	z_4
z_1'	$\cos\theta$	$\sin\theta$	0	0
z_2'	$-\sin\theta$	$\cos\theta$	0	0
z_3'	0	0	$\cos\theta$	$-\sin\theta$
z_4'	0	0	$\sin\theta$	$\cos\theta$

(W_1)

$$z_1 = a_1\cos\theta + a_2\sin\theta, \qquad z_2 = -a_1\sin\theta + a_2\cos\theta,$$
$$z_3 = a_3\cos\theta - a_4\sin\theta, \qquad z_4 = a_3\sin\theta + a_4\cos\theta. \tag{78}$$

The same reasoning as above shows that these lines are parallel to the x_3-axis or the edge A_3A_4. Thus through a given point a there pass two Clifford parallels relative to the x_3-axis.

We wish to show now that these Clifford parallels lie on a certain *torus* whose axis is the x_3-axis. Let P be an arbitrary point of space. Let the meridian plane μ through the x_3-axis and P cut the edge A_1A_2 of the polar tretrahedron $A_1, \cdots, A_4$ we have used so often in the point Q. The coördinates $y_1, \cdots, y_4$ of Q are as we saw in (73):

$$y_1 = R\cos\alpha, \qquad y_2 = R\sin\alpha, \qquad y_3 = y_4 = 0. \tag{79}$$

Thus the equation of the meridian plane μ is

$$z_2 - z_1\tan\alpha = 0 \qquad \text{or} \qquad z_2/z_1 = \tan\alpha; \tag{80}$$

and the coördinates $z_1, \cdots, z_4$ of P lying in this plane must satisfy (80). The R-straight through P, Q cuts the x_3-axis in L, say, and as the edge A_1, A_2 is the reciprocal polar of the x_3-axis, *i.e.*, of the edge A_3A_4, we have $\overline{QL} = \overline{QP} + \overline{PL} = \pi R/2$. If we set $\overline{PL} = \delta$, $\overline{PQ} = d$, this last relation gives $d + \delta = \pi R/2$.

To find the surface on which P lies we have only to remember that the double rotation W rotates the meridian plane μ about the x_3-axis and P moves in this plane so that $\overline{LP} = \delta$ is constant, since all lengths are unchanged by a displacement. We have now, by (55) and (79),

$$\cos\frac{d}{R} = \frac{y_1z_1 + \cdots + y_4z_4}{R^2} = \frac{R(z_1\cos\alpha + z_2\sin\alpha)}{R^2}$$
$$= \frac{Rz_1(\cos\alpha + z_2/z_1\sin\alpha)}{R^2} = \frac{z_1(\cos\alpha + \tan\alpha\sin\alpha)}{R}$$

by (80). Hence

$$\cos\frac{d}{R} = \frac{z_1\sec\alpha}{R} = \sin\frac{\delta}{R} \qquad \text{as} \qquad d + \delta = \frac{\pi R}{2}.$$

Thus

$$z_1^2 \sec^2\alpha = R^2 \sin^2 \delta/R \quad \text{or} \quad R^2 \sin^2 \delta/R = z_1^2 (1 + \tan^2 \alpha)$$

or, using (80),

$$R^2 \sin^2 \delta/R = z_1^2 + z_2^2. \tag{81}$$

Hence replacing z_1, z_2 by their x values as given by (22),

$$(x_1^2 + x_2^2) \frac{16R^4}{(r^2 + 4R^2)^2} = R^2 \sin^2 \frac{\delta}{R}; \qquad r^2 = x_1^2 + x_2^2 + x_3^2.$$

Thus finally the point P moves on the surface

$$(x_1^2 + x_2^2 + x_3^2 + 4R^2)^2 = 16R^2(x_1^2 + x_2^2) \operatorname{cosec}^2 \delta/R. \tag{82}$$

This is a torus in E-geometry, generated by rotating the E-circle

$$(x_1 - 2R \operatorname{cosec} \delta/R)^2 + x_3^2 = 4R^2 \cot^2 \delta/R \tag{83}$$

about the x_3-axis. The coördinates of the center C of this circle are:

$$x_1 = 2R \operatorname{cosec} \delta/R, \qquad x_2 = 0, \qquad x_3 = 0; \tag{84}$$

its radius

$$t = 2R \cot \delta/R. \tag{85}$$

Since every R-straight lies in an E-plane passing through the origin, the Clifford parallels to the x_3-axis are plane sections of this torus. We show that these sections are made by the planes ω which are tangent internally to the torus.

For let P be one of the two points where a Clifford parallel γ meets the fundamental sphere. The meridian plane μ passing through the x_3-axis and P cuts [1] the fundamental sphere in a circle of radius $2R$ in E-measure. It cuts out two circles K, K' from the torus whose centers C, C' lie on a line OM perpendicular to the x_3-axis such that $\overline{OC} = \overline{OC'} = 2R \operatorname{cosec} \delta/R$ in E-measure, by (84), and the radius of each is $t = 2R \cot \delta/R$ by (85). Since $\overline{OC}^2 = \overline{OP}^2 + \overline{PC}^2$, the triangle OPC is a right triangle and OP is tangent to the circle K. Since curves meeting at P and drawn on the torus have their tangents at P lying in the tangent plane to the torus at P, the plane ω passes through OP and is tangent to the torus at P.

Let l, l' be the two Clifford parallels through the point $a = (a_1, \cdots, a_4)$. Giving θ the value $d\theta$ in (75), (78), we get

$$\begin{aligned} dz_1 &= a_2 d\theta, & dz_2 &= -a_1 d\theta, & dz_3 &= a_4 d\theta, & dz_4 &= -a_3 d\theta, & &\text{on } l, \\ \delta z_1 &= a_2 d\theta, & \delta z_2 &= -a_1 d\theta, & \delta z_3 &= -a_4 d\theta, & \delta z_4 &= a_3 d\theta, & &\text{on } l'. \end{aligned}$$

The angle φ between these two arcs of length $ds = \delta s = Rd\theta$ is given by

$$\begin{aligned} \cos \varphi &= \frac{dz_1}{ds} \cdot \frac{\delta z_1}{\delta s} + \cdots + \frac{dz_4}{ds} \cdot \frac{\delta z_4}{\delta s} \\ &= \frac{(a_2^2 + a_1^2 - a_4^2 - a_3^2) d\theta^2}{R^2 d\theta^2} = \frac{a_2^2 + a_1^2 - a_4^2 - a_3^2}{R^2}. \end{aligned} \tag{86}$$

[1] It will aid the reader to draw the figure.

Now the distance δ of a from the x_3-axis is given by (81), *viz.*,

$$\sin^2 \frac{\delta}{R} = \frac{a_1^2 + a_2^2}{R^2} = \cos^2\left(\frac{\pi}{2} - \frac{\delta}{R}\right). \tag{87}$$

As

$$\cos^2 \frac{1}{2}\varphi = \frac{1 + \cos\varphi}{2} = \frac{a_1^2 + a_2^2}{R^2}$$

by (86) we have, comparing this with (87),

$$\cos \frac{1}{2}\varphi = \cos\left(\frac{\pi}{2} - \frac{\delta}{R}\right).$$

Therefore,

$$\frac{1}{2}\varphi = \frac{\pi}{2} - \frac{\delta}{R}.$$

If, instead of φ, we use the supplementary angle $\psi = \pi - \varphi$, we get

$$\psi = 2\delta/R.$$

Since a displacement enables us to make any R-straight coincide with the x_3-axis leaving all distances and angles unaltered, we have the

THEOREM: *Through any point of space P there pass two Clifford parallels l, l' to a given line m. If ψ is the angle between l, l' and $\delta = PQ$ = the R-distance of P to m, $\psi = 2\delta/R$. The R-plane in which l, l' lie is perpendicular to the R-straight PQ, and ψ is bisected by the R-plane through m and P.* The surface on which these parallels lie is called a Clifford surface.

11. Areas and Volumes. These notions we define briefly. Suppose in E-geometry we pass from rectangular coördinates x_1, x_2, x_3 to curvilinear coördinates u_1, u_2, u_3. In the new coördinates

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

becomes

$$ds^2 = \sum a_{ij} du_i du_j; \qquad a_{ij} = a_{ji}, \qquad i, j = 1, 2, 3. \tag{88}$$

We call

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{89}$$

the determinant of the quadratic differential form (88). It is shown in the calculus [1] that the element of volume

$$dv = dx_1 dx_2 dx_3$$

becomes, in terms of the new variables,

$$dv = \sqrt{|a|}\, du_1 du_2 du_3. \tag{90}$$

[1] See, for example, Goursat-Hedrick, *Mathematical Analysis*, vol. 1, Boston, 1904, p. 305.

We shall adopt (90) as our definition of volume in R-geometry where ds^2 is replaced by (5) or one of its equivalent forms, as (10) for example.

Similarly it is shown in the calculus that if

$$ds^2 = a_{11}du_1^2 + 2a_{12}du_1du_2 + a_{22}du_2^2$$
$$= \sum a_{ij}du_idu_j, \qquad a_{ij} = a_{ji}, \qquad i, j = 1, 2, \tag{91}$$

defines the element of arc on a surface S, then

$$dA = \sqrt{|a|}\,du_1du_2 \tag{92}$$

defines the element of area on S. Here a is the determinant of (91) or

$$a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{93}$$

We adopt (92) as our definition of area in R-geometry, the form (91) being now expressed in R-measure.

An important and very necessary property of these definitions of area and volume is that they are unaltered after a displacement; that is to say, if for example v is the volume of a body B, let B be moved into the position B'. If v' is the volume of B', then $v = v'$.

To illustrate these definitions let us find the area and volume of an R-sphere. This we define as the locus of all points P, whose distance in R-measure from a fixed point C is constant. This distance $\overline{PC} = \rho$, say, is of course measured along the R-straight PC; we call PC a radius.

Without loss of generality, we may suppose the center C is the origin and ds is given by (10). Then

$$a = \begin{vmatrix} 1 & 0 & 0 \\ 0 & R^2\sin^2\rho/R & 0 \\ 0 & 0 & R^2\sin^2\rho/R\sin^2\theta \end{vmatrix}.$$

Therefore

$$\sqrt{a} = R^2\sin^2\frac{\rho}{R}\sin\theta.$$

Thus

$$v = R^2\int_0^{\rho}\sin^2\frac{\rho}{R}d\rho\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi$$
$$= 4\pi R^3\int_0^{\rho}\sin^2\frac{\rho}{R}\cdot d\left(\frac{\rho}{R}\right) = 2\pi R^2\rho - \pi R^3\sin\frac{2\rho}{R}. \tag{94}$$

The radius of the fundamental sphere F in R-measure is $\rho = \pi R/2$. This in (94) gives as the volume of all R^* space the following:

$$V_F = \pi^2R^3. \tag{95}$$

For $\rho = \pi R$, (94) becomes

$$V_\infty = 2\pi^2R^3, \tag{96}$$

the volume of all R-space.

To find the area of an R-sphere of radius ρ, we may suppose as before that its center is at O. As ρ = constant on this sphere S, the element of arc ds on S is obtained by setting $d\rho = 0$ in (10), which gives

$$ds^2 = R^2 \sin^2 \frac{\rho}{R}(d\theta^2 + \sin^2 \theta d\varphi^2).$$

This takes the place of (91); hence

$$a = \begin{vmatrix} R^2 \sin^2 \rho/R & 0 \\ 0 & R^2 \sin^2 \rho/R \sin^2 \theta \end{vmatrix}$$

and

$$A = R^2 \sin^2 \frac{\rho}{R} \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi = 4\pi R^2 \sin^2 \frac{\rho}{R}. \tag{97}$$

Let S be an R-sphere of radius ρ in R-measure and center C. We show that S is also an E-sphere whose center of course is not C in general. For let $z_1, \cdots z_4$ be the coördinates of a point P of S and let $A_1, \cdots A_4$ be the coördinates of its center C. Then

$$\cos \frac{\rho}{R} = \frac{A_1 z_1 + \cdots + A_4 z_4}{R^2} = k,$$

where k is a constant since ρ is constant. Hence, setting $-kR^2 = B$,

$$A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4 + B = 0.$$

Replacing the z's by the x's, using (22) and (23), we get

$$4R^2(A_1 x_1 + A_2 x_2 + A_3 x_3) + A_4(4R^2 - r^2) + B(r^2 + 4R^2) = 0$$

or

$$(x_1^2 + x_2^2 + x_3^2)(B - A_4) + 4R^2(A_1 x_1 + A_2 x_2 + A_3 x_3) + 4R^2(B + A_4) = 0$$

which is an E-sphere.

Let us define a circle in R-geometry as the locus of all points in an R-plane at a constant distance from a fixed point in this plane. The foregoing shows that R-circles are also E-circles with different center in general. The intersection of two R-spheres or the intersection of an R-plane with an R-sphere is an R-circle.

The length in R-measure of an R-circle of radius ρ is

$$s = 2\pi R \sin \rho/R$$

and its area is by (92)

$$A = 2\pi R^2(1 - \cos \rho/R).$$

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. THE CONSEQUENCES OF ROLLE'S THEOREM.

By A. A. BENNETT, University of Texas.

Rolle's Theorem may be stated as follows:

If $F(x)$ be defined as a one-valued function on the interval[1] $a \leq x \leq b$, *and* (1) *vanishes at* $x = a$ and $x = b$, (2) *is continuous from within the interval at* a *and at* b, (3) *has a derivative at every point of the segment* a, b; *then there exists a point* ξ *of the segment such that* $F'(\xi) = 0$. We will assume that this theorem has been established in the usual manner.

Among the numerous important consequences of this theorem that have been enumerated there are two involving only first derivatives, that are given in most texts on the calculus. One due to Lagrange and given in all standard modern texts states that under suitable conditions

$$\frac{f(b) - f(a)}{b - a} = f'(\xi), \ \xi \text{ being some point between } a \text{ and } b.$$

The other, due to Cauchy, and somewhat less extensively quoted, states that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}, \ \xi \text{ as before.}$$

The former, on account of its extensive utility, is often called the "law of the mean," or "mean value theorem," the latter being styled, in contrast, the "second law of the mean," or "extended mean value theorem." These names have the advantage of avoiding reference to the discoverers or reputed discoverers, since the ascribing to historical personages is usually controversial, often misleading, of little educational advantage, and of no logical import. On the other hand, these particular terms are unsatisfactory, since many writers call the second of these extensions the "theorem of the mean" and prove it first.

Even the clearest texts do not hesitate in this connection to introduce apparently artificial functions which can be identified with the F of Rolle's theorem. These functions are justified by the fact that they serve to establish the desired theorem as a consequence of a known theorem, but are not psychologically motivated. The result is that the student is mystified, and the particular form of the function is difficult to keep in mind. There is no need of this artificiality, and in the case of Lagrange's theorem, there seems to be no excuse for it. When

[1] The term *interval* will be used to include the end-points, assumed distinct, while *segment* refers to the same set exclusive of end-points.

Lagrange's theorem of the mean is proved independently of Cauchy's, the function used for F is usually although not universally that given by the following obvious geometrical consideration. The ordinate difference between the curve and the chord (which becomes the ordinate under a simple transformation of axes) obviously satisfies the conditions required for F, and is therefore used. Osgood mentions this fact but there is scarcely a single foreign treatise that hints at the meaning of the function used. In the case of Cauchy's theorem, silence seems unanimous. This is the more surprising since this may be given the same geometrical interpretation as before, by using the independent variable as a parameter. If $f(t)$ be called y, and $g(t)$ be called x, Cauchy's theorem like Lagrange's (although now applicable to a wider class of curves) states that there exists a point on the arc for which the tangent is parallel to the chord.

The proof of Cauchy's theorem is not materially simplified by first proving Lagrange's, although Lagrange's is an immediate corollary of Cauchy's if $g(x)$ be identified with x. If both theorems are to be demonstrated, the total space devoted to proof will be reduced if Cauchy's be proved from Rolle's theorem, and Lagrange's be inferred as a corollary. Since the hypothesis of Rolle's theorem is much more restrictive than that of Cauchy's, some new functions must be introduced. We may either state the theorem to be proved, and then seek to relate it to Rolle's theorem, or we may set up a simple function which obviously satisfies Rolle's theorem and as obviously leads to Cauchy's. We shall do the former, but by reversing the steps we would follow the latter course.

We may write the conclusion of Cauchy's theorem in the following symmetrical algebraic form: There exists at least one value, ξ, between a and b such that

$$\begin{vmatrix} x'(\xi) & y'(\xi) & 0 \\ x(a) & y(a) & 1 \\ x(b) & y(b) & 1 \end{vmatrix} = 0. \tag{1}$$

This[1] may be compared with the more usual form and identified with it, or may be directly verified from the first principles of the analytic geometry of plane vectors as the condition that the infinitesimal vector from the origin to $[x'(\xi), y'(\xi)]$ shall be parallel to the line joining $[x(a), y(a)]$ and $[x(b), y(b)]$. It is at once obvious that (1) is of the form $F'(\xi) = 0$, where $F(t)$ satisfies the condition

$$F(t) \equiv \begin{vmatrix} x(t) & y(t) & 1 \\ x(a) & y(a) & 1 \\ x(b) & y(b) & 1 \end{vmatrix}. \tag{2}$$

This in turn is on the face of things one of the simplest possible functions involving the two arbitrary functions $x(t)$ and $y(t)$, which will vanish for $t = a$ and $t = b$, as desired for Rolle's theorem.

It is clear from the form of the proof that in place of (2) we might have

[1] Vivanti, *Analisi Infinitesimale*, page 100, uses this expression.

taken[1]

$$F(t) \equiv \begin{vmatrix} x(t) & y(t) & z(t) \\ x(a) & y(a) & z(a) \\ x(b) & y(b) & z(b) \end{vmatrix}, \tag{3}$$

or even

$$F(t) \equiv \begin{vmatrix} x(t) & y(t) & z(t) & 1 \\ x(a) & y(a) & z(a) & 1 \\ x(b) & y(b) & z(b) & 1 \\ A & B & C & 1 \end{vmatrix}. \tag{4}$$

From (4) we obtain the geometric theorem that there is at least one point on the general space arc considered at which the tangent line is parallel to the plane through (A, B, C) and the chord, the point (A, B, C) being any given point not on the line of the chord. From (3) we have the special case in which (A, B, C) is the origin.

II. Note on the Comparison of Aggregates.

By H. L. Slobin, University of New Hampshire.

Professor R. S. Underwood in his article in the Monthly (*1922*, 346) supplements a theorem with the words "The above theorem is interesting in that it brings out in a striking way for pedagogical purposes the great preponderance of irrational over rational numbers."

It is obvious that in any continuous interval there is a correspondence between the aggregate of the values of the argument of the trigonometric functions and the aggregate of the values of the functions themselves. The values of the functions are of two classes, the rational and the irrational. The irrational are of two classes, the algebraic irrational and the transcendental irrational. The irrationals determined by Professor Underwood are of the algebraic irrational class only. The method of comparing two aggregates is by their cardinal number; *i.e.*, the power or potency. The algebraic numbers are enumerable, and in particular the algebraic irrationals are enumerable, and the rational numbers are enumerable, and hence the aggregate of rational numbers and the aggregate of algebraic irrational numbers are equivalent. The cardinal number of the irrational or of the transcendental numbers in any interval is C, while that of an enumerable set is $\aleph_0$; where $C > \aleph_0$. It may indeed be permissible to speak in general of the "preponderance of irrational numbers over rational numbers," but I do not think it permissible or advisable thus to speak in comparing two enumerable aggregates such as those irrationals considered by Professor Underwood.

[1] Epstein in Pascal's *Repertorium*, 2d ed., page 469, gives (3).

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, Ithaca, N. Y., to whom communications should be sent.

REVIEWS.

Vector Analysis and the Theory of Relativity. By F. D. MURNAGHAN. Baltimore, The Johns Hopkins Press, 1922. 8vo. x + 125 pp. Price $2.75.

The statement has commonly been attributed to Einstein at the time that he first published the generalized theory of relativity that there were only twelve people in the world who were capable of understanding his paper. If such a statement was made, it was of course true only in the sense that at that particular time very few scholars were familiar with all the physical and mathematical theories employed in the paper, and not at all in the sense that the relativity theory is so abstruse that it must remain beyond the reach of the majority of physicists and mathematicians. In particular, Einstein made use of the "absolute differential calculus" which had been developed by Ricci and Levi-Civita about fifteen years earlier, but which was little known outside of a narrow circle of mathematical specialists.

As soon as the importance of the generalized theory of relativity was realized, and especially after the prestige it acquired from the verification in 1919 of its prediction of the amount of deviation of a light-ray passing near the sun, a demand arose for an exposition which would make the details of the theory accessible to physicists of average training. The present book, which is based on a series of lectures delivered by the author to graduate students at Johns Hopkins, is an attempt to meet this need. An introductory note by Professor Joseph S. Ames testifies to its success in doing so.

In the first chapter the concept of a tensor is introduced by the aid of line and surface integrals; it is shown that for an integral along a curve to have a value independent of the particular system of coördinates used, the functions in the integrand must be the components of a covariant vector or tensor of the first rank, and similarly in a surface integral the functions must form a covariant tensor of the second rank, etc. The general definitions of covariant, contravariant, and mixed tensors are then given by means of their transformation equations.

The second chapter is devoted to the algebra of tensors. The importance of tensors is due to the fact that the vanishing of a tensor (or the equality of two tensors) is a property which is independent of the coördinate system used. The generalized theory of relativity regards all systems of coördinates as on the same footing, and aims to express the laws of physics in a form which will be the same for all observers, regardless of their relative motions. This object is attained if they can be expressed by the vanishing of tensors.

In chapters III and IV the further theory of tensors is developed, and its character as a generalization of the ordinary vector analysis adapted to all types of space and to all coördinate systems is brought out. The latter part of chapter IV and most of chapter V are devoted to applications of the tensor analysis to certain problems in electromagnetic theory.

In chapter VI the absolute differential calculus is taken up, leading to the introduction of the tensor of the second rank whose vanishing gives Einstein's law of gravitation. The chapter ends with a discussion of the curvature of space. In the seventh and last chapter the preceding analysis is applied to the classical problems of relativity, including the metrical character of the space near a heavy body, the motion of a particle under Einstein's law of gravitation, with special reference to Mercury, and the deviation of a light-ray in a gravitational field.

Misprints and minor discrepancies of notation are rather numerous, but most of them are not such as to cause any difficulty to the reader. Apparently no serious attempt was made to secure consistency in such matters as enclosing superscripts in parentheses or using commas between the components of multiple subscripts, and the squares of numbers with superscripts are printed in three or four different ways.

The subject-matter of the book is rather abstract, and it is by no means easy reading. This is, however, largely inherent in the nature of the topics treated; as Eddington says, "I doubt if there is any royal road to relativity, and it is scarcely possible to make serious progress except by analytic methods."

It is an interesting feature of the history of physics that the mathematical framework of a theory often outlives the theory itself. Thus much of the mathematics of the old theory that light consists of mechanical vibrations in an elastic medium called the ether was taken over by the electromagnetic theory; and if the latter theory is to be displaced in the near future by a new one in which the concept of the ether is entirely dispensed with, as seems quite possible, we may feel sure that Maxwell's equations will continue to play a prominent part. Similarly the mathematics which has been developed in connection with Einstein's theory will probably prove to be a permanent contribution to mathematical physics, whatever may be the fate of the physical hypotheses on which that theory is now based.

P. M. BATCHELDER.

James Stirling: A Sketch of his Life and Works, along with his Scientific Correspondence. By CHARLES TWEEDIE. Oxford, Clarendon Press, 1922. 10 + 213 pp. Price $5.35 (in England, 16 s.).

We have here, set in its background of politics, travel, commerce and science, an entertaining account of the life and achievements of James Stirling, a mathematician of no mean ability, but overshadowed by such contemporaries as his friends Newton, Euler, Bernoulli, De Moivre, Cramer, and others. In addition to a spirited biography enriched with numerous extracts from private letters, this book gives a judicious summary of his now rare published works: (A) "Enumeration of Cubics," (B) "Methodus Differentialis," (C) "Contributions to the *Philosophical Transactions*." These summaries are handled in an intelligent and efficient manner, showing a mathematical comprehension of the material that is delightful as it is unusual in a biographer. The last third of the book is

devoted to the publication of a collection of Stirling's scientific correspondence, enlightened by not a few explanatory notes by the author.

Stirling is best known for his asymptotic formula in connection with the Gamma Function, but this special result is merely characteristic of his bold and ingenious treatment of many rather intractable analytical expansions, and his recourse to difference equations. While the modern mathematician will not be as indulgent as he in questions of rigor, Stirling's originality during his brief productive period sets a high standard for modern progress. It is impossible here to summarize his wide range of activities but mention might be made in passing of his geometrical work on cubic curves, his study of series, and his investigation of the shape of the earth, all questions suggestive of and perhaps suggested by his friend and counsellor, Sir Isaac Newton.

As a strictly mathematical reference book this work must remain among those of secondary importance. It has not the comprehensiveness of a complete edition of collected works, and on the other hand most of the mathematical problems here published for the first time are no longer in the forefront of mathematical investigation. Many minor points of considerable interest and some importance in the history of mathematics and in the biography of mathematicians receive fresh light from the careful antiquarian research of the author. Despite a few misprints and other trivial objections, this volume is thoroughly to be commended in its purpose, its use of material and its delightful spirit. Its appeal is indeed a wide one, and its glimpse of the heroic days of the early eighteenth century will be suggestive and entertaining to many who will not have Stirling's ambition in mathematical research.

A. A. BENNETT.

Vorlesungen über die Theorie der algebraischen Zahlen. By E. HECKE. Leipzig, Akademische Verlagsgesellschaft, 1923. Paper, 8vo. viii + 266 pages. Price (bound) 14 Swiss francs.

In view of Hecke's important recent researches in the analytic theory of numbers, his book is certain to attract special attention from all workers in this field. Its subject matter has been so well selected, many of the proofs have been so materially simplified, and the presentation is so clear and attractive that the book is recommended without reservation to those seeking an adequate introduction to the theory of algebraic numbers as developed to date. It was possible to greatly shorten and simplify numerous proofs and to refer them to a common source by employing properties of finite and infinite abelian groups established in 28 pages in Chapter II, and applied in Chapter III to develop needed theorems in ordinary elementary theory of numbers.

Chapters IV and V devote 94 pages to the classical theory of the algebra and arithmetic of algebraic fields. Chapter VI presents in 18 pages the determination by the transcendental methods of Dirichlet and Dedekind of the number of classes of ideals and theorems on the distribution of prime ideals, including a proof of the existence of an infinitude of rational primes in any arithmetical

progression. Chapter VII devotes 45 pages to quadratic fields, including determinations of the number of classes of ideals with and without the use of zeta functions. The final Chapter, VIII, presents a new proof of the general quadratic reciprocity law in an arbitrary algebraic field. This proof, which is much shorter than all previous proofs, employs Gauss sums extended to any field and theta functions with n variables.

In 1918 Landau published his brief introduction to the elementary and analytic theories of algebraic numbers. This was followed by a larger book along similar lines by Bianchi. One of these books and Hecke's new book should certainly be in the hands of any one desiring to acquaint himself with the theory of algebraic numbers to date. That theory is so fundamental that no mathematician can afford to remain ignorant of it. The book by Hecke presupposes no acquaintance whatever with the theory of numbers and leads the reader by the shortest routes to the chief results to date concerning algebraic numbers. So valuable a book should be accessible to every mathematician.

L. E. DICKSON.

Geometry of Greek Vases. By L. D. CASKEY. Boston, Museum of Fine Arts, 1922. 4to. 11 + 235 pages. Price $6.00.

This book is to be regarded as a supplement to Jay Hambidge's *Dynamic Symmetry: the Greek Vase,* which was reviewed by Professor A. A. Bennett in this MONTHLY (*1922*, 169–171). Mr. Caskey has made careful measurements of practically all the examples of Attic pottery in the collection of the Boston Museum, and the book contains these together with an analysis of the way in which they fit into the "dynamic symmetry" plan of construction. The Introduction (pp. 1–34) summarizes the results, and contains all that is of interest from a mathematical viewpoint. It is found that in 195 measurements the departure from the dimensions computed according to a "dynamic" analysis amounts to less than 2 mm., while in 68 instances the discrepancy is over 2 mm. While this result is not so striking as to afford a substantial ground for accepting Hambidge's theory of "dynamic symmetry," it does indicate that in a large number of cases such a geometrical construction is a very plausible explanation of the design employed by the Greek artists. Indeed, when we consider the fondness that the Greeks constantly showed for geometric constructions in general, and, since the days of the early Pythagoreans, for the regular pentagon in particular, it does not seem at all idle fancy, still less mere "ingenious magic" as one writer has put it,[1] to imagine that geometric constructions of the simple sort indicated (ratios of $\sqrt{2}$, $\sqrt{5}$, and $(\sqrt{5}-1)/2$ for the most part) were actually employed as the basis for the proportions of the parts of many Attic vases.

The book is well printed, and the numerous drawings carefully and attractively reproduced.

R. B. MCCLENON.

[1] R. Carpenter in *American Journal of Archaeology*, 25 (1921), 18–36.

ARTICLES IN CURRENT PERIODICALS

AMERICAN JOURNAL OF MATHEMATICS, volume 45, April, 1923: "A class of numbers connected with partitions" by E. T. Bell, 73–82; "Note on a new type of summability" by N. Wiener, 83–86; "On mediate cardinals" by Dorothy Wrinch, 87–92; "Periodic oscillations of three finite masses about the Lagrangian circular solutions" by H. E. Buchanan, 93–121; "On certain chains of theorems in reflexive geometry" by Flora D. Sutton, 122–144; "A poristic system of equations" by L. B. Robinson, 145–153.

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BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 29, July, 1923: "Report on continuous curves from the viewpoint of analysis situs" by R. L. Moore, 289–302; "A generalization of a property of an acnodal cubic curve" by H. Hilton, 303–308; "On uniform limitedness of sets of functional operations" by T. H. Hildebrandt, 309–315; "Beck on coördinate geometry" [review of H. Beck, *Koordinaten-Geometrie, I. Die Ebene* (Berlin, 1919)] by J. L. Coolidge, 316–318; "Keynes on probability" [review of J. M. Keynes, *A Treatise on Probability* (London, 1921)] by E. B. Wilson, 319–322; "Blaschke on differential geometry" [review of W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie* (Vol. I, Berlin, 1921)] by G. A. Bliss, 322–325; Reviews: by C. N. Moore of H. S. Carslaw, *Introduction to the Mathematical Theory of the Conduction of Heat in Solids* (2nd ed., London, 1921), 326–327; by E. Swift of C. Runge, *Praxis der Gleichungen* (2nd ed., Berlin, 1921), 327–328; by P. J. Daniell of A. E. Kennelly, *Les Applications élémentaires des Fonctions Hyperboliques à la Science de l'Ingénieur Electricien* (Paris, 1922), 238; and by J. B. Shaw of J. Boussinesq, *Cours de Physique Mathématique, Compléments au Tome III* (Paris, 1922), 329–330; Notes, 331–333; New publications, 334–336.—October: "On the Riemann Zeta function" by C. F. Craig, 337–340; "Note on a certain type of ruled surface" by W. C. Graustein, 341–344; "The differentiation of a function of a function" by H. S. Carslaw, 344; "On the relative curvature of two curves in V_n" by J. Lipka, 345–348; "Square partition congruences" by E. T. Bell, 349–355; "Determination of all systems of ∞^4 curves in space in which the sum of the angles of every triangle is two right angles" by J. Douglas, 356–366; "Volume III of Lie's Memoirs" [review] by R. D. Carmichael, 367–369; "Bolzano on paradoxes" [review of B. Bolzano, *Paradoxien des Unendlichen* (Leipzig, 1921)] by L. L. Silverman, 370–371; Reviews: by G. A. Miller of A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung, mit Anwendungen auf algebraische Zahlen und Gleichungen sowie auf Kristallographie* (Berlin, 1923), 372; by C. H. Sisam of E. Pascal, *Repertorium der höheren Mathematik* (2nd ed., Vol. II, part 2, Geometry of Space, Leipzig, 1922), 373; by C. H. Forsyth of H. Andoyer, *Tables logarithmiques à treize décimales* (Paris, 1922), 373, and of J. W. Glover, *Tables of Applied Mathematics in Finance, Insurance, Statistics* (Ann Arbor, 1923), 376–379; by D. N. Lehmer of A. Cunningham, *Fundamental Congruence Solutions* (London, 1923) and *Haupt-exponents, Residue-indices, Primitive Roots, and Standard Congruences* (London, 1922), 374–375; by H. B. Phillips of J. B. Shaw, *Vector Calculus* (New York, 1922), 375; by F. Cajori of Jamblichus, *Theologoumena Arithmetica* (Leipzig, 1922), 377; and by J. B. Shaw of S. Valentiner, *Vektoranalysis* (3rd ed., Berlin, 1923), 377; Notes, 378–380; New publications, 381–384.

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JOURNAL FÜR DIE REINE UND ANGEWANDTE MATHEMATIK (CRELLE), volume 153, nos. 1–2, August, 1923: "Ein- und Zweischaligkeit von Flächen 3. Ordnung" by R. Sturm, 1–7; "Arithmetische Eigenschaften der Polynomialkoeffizienten" by K. Hensel, 8–11; "Symmetrische Matrizen im Körper der rationalen Zahlen" by H. Hasse, 12–43; "Beiträge zur Theorie der

Laguerreschen Polynome und zum Summationsproblem von Orthogonalsystemen" by G. Wiarda, 44–60; "Bemerkung zu Herrn A. Fraenkel's Aufsatz 'Die Berechnung des Osterfestes'" by K. Boecklen, 61–65; "Ueber die dreifach ausgedehnte Mannigfaltigkeit der ebenen Dreiecke mit reellen Seiten" by H. Wolff, 66–75; "Zur Theorie des quadratischen Hilbertschen Normenrestsymbols in algebraischen Körpern" by H. Hasse, 76–93; "Zur Bewertungstheorie der algebraischen Körper" by K. Rychlik, 94–107; "Bemerkungen zu Herrn Hensels Arbeit 'Die Zerlegung der Primteiler eines beliebigen Zahlkörpers in einem auflösbaren Oberkörper'" by T. Rella, 108–110; "Zur Newtonschen Approximationsmethode in der Theorie der p-adischen Gleichungswurzeln" by T. Rella, 111–112; "Darstellbarkeit von Zahlen durch quadratische Formen in einem beliebigen algebraischen Zahlkörper" by H. Hasse, 113–130; "Ueber die Zerlegung der endlichen Gruppen in direkte unzerlegbare Faktoren" by R. Remak, 131–140.

Mathematische Zeitschrift, volume 17, nos. 3–4, August, 1923: "Ueber die Einordnung der Affingeometrie in die Theorie der höheren Uebertragungen. I" by J. A. Schouten, 161–182; "Ueber die Einordnung der Affingeometrie in die Theorie der höheren Uebertragungen. II" by J. A. Schouten, 183–188; "Die Airysche Funktion für den Ellipsenring" by A. Timpe, 189–205; "Ueber die Picardschen Ausnahmewerte sukzessiver Derivierten" by W. Saxer, 206–227; "Ueber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten" by M. Fekete, 228–249; "Zahlentheoretische Abschätzungen mit Anwendung auf Gitterpunktprobleme" by J. G. van der Corput, 250–259; "Ueber Bildschranken bei Potenzreihen und ihren Abschnitten" by W. Rogosinski, 260–276; "Ueber einen zahlentheoretischen Satz von Hurwitz" by W. Jänichen, 277–292; "Zur Charakterisierung der Drehungsgruppe" by H. Weyl, 293–320.

Messenger of Mathematics, volume 52, nos. 11–12, March and April, 1923: "On Pellian chains" by A. Cunningham, 161–167; "On symmetrical plane algebraic curves" by H. Hilton, 168–175; "A suggestion for a new symbolic treatment of probability" by I. M. Horobin, 176–181; "On a spherical configuration of eight points" by W. Burnside, 181–184; "On the zeros of an integral function represented by Fourier's integral" by G. Pólya, 185–188; "Three n-dimensionals" by F. C. Pitt-Bazett, 189–192.

Proceedings of the National Academy of Sciences of the U. S. A., volume 9, July, 1923: "Form of the numbers of the subgroups of a prime power group" by G. A. Miller, 237–238.—August: "The Einstein equations of the gravitational field for an arbitrary distribution of matter" by T. Y. Thomas, 275–278.—September: "Groups of order 2^m in which the number of the subgroups of at least one order is of the form $1 + 4k$" by G. A. Miller, 326–328.

Scientific Monthly, volume 17, September, 1923: "Inaccuracies in the mathematical literature" by G. A. Miller, 216–228; "A survey of mathematics and astronomy" by E. D. Roe, Jr., 245–254.—November: "Mathematics as a career" by C. J. Keyser, 489–497.

Transactions of the American Mathematical Society, volume 25, January, 1923: "The (1, 1) correspondence associated with the cubic space involution of order two" by F. R. Sharpe and V. Snyder, 1–12; "Sur certaines équations aux différences finies" by N. E. Norlund, 13–98; "Differential geometry of an m-dimensional manifold in a Euclidean space of n dimensions" by C. E. Wilder, 99–122; "Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier series expansions" by C. C. Camp, 123–134; "Euler algebra" by E. T. Bell, 135–154.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 54, no. 3, published July 20, 1923: "Der mathematische Unterricht und der Bergbau" by F. Requard, 129–132; "Ueber die Sichtung des mathematischen Unterrichtsstoffes in den höheren Lehranstalten" by R. Fuchs, 132–138; "Experimentelle Geometrie der Richtung ausgeführt für die Ebene" by K. Bochow, 139–148; "Die Zahl e im Unterricht" by A. Harnack, 148–152; "Beiträge zur Behandlung der Kombinatorik" by W. König, 152–154; "Die Schnittpunkte zweier Kreise" by A. Witting, 155–156; "Die Behandlung einiger Grundbegriffe der Mechanik im Schulunterricht" by F. Hund, 156–162; "Die Verwendung der Glimmlampe im Unterricht" by A. Lindemann, 162–168; "Die Entstehung der Brennpunkte im Kugelschatten" by A. Larmer, 168–169; "Die Lehre von der Chordale in lagengeometrischer Behandlung" by R. Roth, 169–170; "Leonhard Eulers Stellung zur Relativität" by E. Hoppe, 181–184.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL AND H. L. OLSON.

Send communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3035 [1923, 337]. Proposed by R. M. MATHEWS, Wesleyan University.

This problem was incorrectly stated. It should read: Generalize projectively and prove that the envelope of the bisectors of the angles between corresponding lines of two perspective pencils is a curve of the *third* class.

3050. Proposed by C. N. MILLS, State Normal, Aberdeen, South Dakota.

Eliminate x, y, z from the equations

$$x^4/a^{5/4}b^{3/4} + y^4/a^{3/4}b^{5/4} = mz^2,$$
$$x^{3/2}/(az)^{1/4} = x + y = y^{3/2}/(bz)^{1/4}$$

and show that, if $ab > 0$, m cannot be less than 2^9.

3051. Proposed by NORMAN ANNING, University of Michigan.

Given the sequence: $u_1 = 2$, $u_2 = 8$, $u_n = 4u_{n-1} - u_{n-2}$ $(n = 3, 4, 5, \cdots)$, show that

$$\frac{\pi}{12} = \sum_{n=1}^{n=\infty} \text{arc cot } u_n^2.$$

3052. Proposed by DR. JOSEF LEWAMDONSKI, Pfaffsatten, Austria.

The ellipse whose parametric equations are:

$$x = a \cos \varphi, \qquad y = b \sin \varphi \tag{1}$$

intersects the conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{2}$$

If φ_1, φ_2, φ_3, φ_4 are the eccentric angles of the four points of intersection, prove that

$$\tan \tfrac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = abB/(a^2A - b^2C).$$

3053. Proposed by DR. JOSEF LENSE, University of Vienna, Austria.

Let $a_1 = a$, $a_2 = a^{a_1}$, $a_3 = a^{a_2}$, $\cdots$, $a_{n+1} = a^{a_n}$, $\cdots$. Discuss the $\lim_{n=\infty} a_n$ as a function of a.

(The limit exists for some values of $a > 1$.)

3054. Proposed by S. A. COREY, Des Moines, Iowa.

Construct three triangles with vector sides a, b, c; d, e, f; g, h, k, such that $d = a$, $g = 2a$, $e = 2b$, $h = b$. Prove that $a^2 + b^2 + 4c^2 = f^2 + k^2$.

SOLUTIONS.

2936 [1921, 467; 1923, 339–341]. Proposed by J. P. BALLANTINE, Columbia University.

A person in drinking from a conical drinking glass tips it at a constant angular rate. At what angle will the delivery be the maximum and at what angle will the surface of the water be a maximum.

CORRECTION BY OTTO DUNKEL, Washington University.

In the solution of problem **2936** (*1923*, 340, 341) the reasoning on page 341 should be replaced by the following:

The greatest delivery is given by the acute angle φ determined from the equation above, provided $1 > \cos 2\alpha > 1/3$. For the derivative of $\cos(\varphi + \alpha)\sin^2(\varphi + \alpha)/\cos^3(\varphi - \alpha)$, multiplied by a factor which is positive in the range $0 \leqq \varphi \leqq \pi/2 - \alpha$, is equal to

$$f(\varphi) = 3\cos(\varphi + \alpha)\cos 2\alpha - \cos(\varphi - \alpha)$$

and, in this case, $f(\varphi)$ is positive from $\varphi = 0$ until it vanishes for the above-mentioned acute angle φ, and then it becomes negative.

In every other case the greatest delivery is given by $\varphi = 0$; for if $\cos 2\alpha \leqq 1/3$, then $f(\varphi) \leqq -2\sin\varphi\sin\alpha$ and hence it is never positive in the range for φ. Therefore the delivery decreases throughout this range. (*End of correction.*)

It is worth noting that the long algebraic inequality on page 340 can be reduced to

$$(1 - x)^2(1 + x)^3(4x^3 + 2x - 7)[(2x + 1)^2 - 8] < 0$$

which is satisfied, in the given range, by $\cos 2\alpha > (2\sqrt{2} - 1)/2 = 0.91421$.

2986 [1922, 356]. Proposed by C. F. GUMMER, Queen's University.

A triangle is inscribed in a circle. The arcs into which it divides the circumference are bisected at points forming the vertices of a second triangle. A third triangle is derived in the same way from the second, and so on. Prove that each set of alternate triangles approaches a limiting position.

Solution by B. F. Yanney, College of Wooster.

We shall first prove that, regardless of position, the limiting triangles of the two sets are equilateral. Let $\theta_1^{(n)}$, $\theta_2^{(n)}$, $\theta_3^{(n)}$ be the arcs into which the vertices of the nth triangle divide the circumference and suppose that they are arranged in descending order of magnitude $\theta_1^{(n)} \geqq \theta_2^{(n)} \geqq \theta_3^{(n)}$; then the next set is $\theta_1^{(n+1)} = (\theta_1^{(n)} + \theta_2^{(n)})/2$, $\theta_2^{(n+1)} = (\theta_1^{(n)} + \theta_3^{(n)})/2$, $\theta_3^{(n+1)} = (\theta_2^{(n)} + \theta_3^{(n)})/2$ and we have $\theta_1^{(n)} \geqq \theta_1^{(n+1)} \geqq \theta_2^{(n+1)} \geqq \theta_3^{(n+1)} \geqq \theta_3^{(n)}$. Also $\theta_1^{(n+1)} - \theta_3^{(n+1)} = (\theta_1^{(n)} - \theta_3^{(n)})/2$. This shows that $\theta_1^{(n)}$ and $\theta_3^{(n)}$ approach limits and that these limits are equal. It then follows that $\theta_2^{(n)}$ approaches the same limit which must be $2\pi/3$, if we suppose that the radius of the circle is unity.

Let δ be a small number less than $\pi/12$ and choose N so large that $|\theta_i^{(N)} - 2\pi/3| < \delta/2$, for $i = 1, 2, 3$. Let A be a vertex of the Nth triangle and mark off on the circle the points A, C', B, A', C, B' at intervals of $\pi/3$. Take these points as mid-points of intervals of length 2δ on the circumference. Then the vertices of the Nth triangle will lie respectively in the A, B, C intervals. Since the vertices of the next triangle are obtained by bisecting the arcs for the Nth triangle, the vertices of the $(N + 1)$th triangle will lie in the A', B', C' intervals. The vertices of all following triangles will fall alternately in these two sets of intervals. By choosing smaller values of δ we obtain in the case of each interval a nest of intervals each lying within the previous interval. Hence in each of the intervals A, B, C there is a limit point, α, β, γ respectively; similarly in A', B', C' there are limit points α', β', γ', respectively. By choosing N large enough we obtain a triangle whose vertices lie as near α, β, γ as we please and at the same time such that the subtended arcs are as near $2\pi/3$ as we please. Hence α, β, γ are the vertices of an equilateral triangle. The vertices of the triangles obtained by bisection from the first set must approach the mid-points of the arcs $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ and, since they also approach α', β', γ', these latter points must be the mid-points.

Also solved by T. Bennett, A. Pelletier and J. Rosenbaum.

2990 [1922, 356]. Proposed by R. M. MATHEWS, Wesleyan University.

If a circle be bitangent to a conic, its center is on one of the axes of the curve.

Solution by Elijah Swift, University of Vermont.

Take axes with the origin at the point of tangency of the two curves and the X-axis tangent to both. Let the radius of the circle be the unit. Then the equation of the circle will be

$$x^2 + (y - 1)^2 = 1 \tag{1}$$

and that of the conic may be written as

$$Ax^2 + 2Bxy + Cy^2 + 2Ey = 0, \tag{2}$$

the term in x and the constant term dropping out on account of the choice of axes.

Two curves are said to have contact of the nth order at a point, if the values of y and the derivatives of y up to and including the nth are equal, respectively, at the point. In this case the contact is of the third order, so that at the origin the values of y, y' $(= dy/dx)$, y'' $(= d^2y/dx^2)$ and y''' $(= d^3y/dx^3)$ must be the same for the two curves.

For points on the circle near the origin we write

$$y = 1 - \sqrt{1 - x^2} = x^2/2 + x^4/8 + \cdots;$$

and this expansion gives, for $x = 0$, $y = 0$, $y' = 0$, $y'' = 1$, $y''' = 0$.

For the conic, take equation (2) and differentiate it three times. The values above (for $x = 0$) must satisfy the resulting equations, which are

$$\begin{aligned} Ax^2 + 2Bxy + Cy^2 + 2Ey &= 0, \\ 2Ax + 2B(xy' + y) + 2Cyy' + 2Ey' &= 0, \\ 2A + 2B(xy'' + 2y') + 2C(yy'' + y'^2) + 2Ey'' &= 0, \end{aligned}$$

and

$$2B(xy''' + 3y'') + 2C(yy''' + 3y'y'') + 2Ey''' = 0.$$

Substituting, we find from the last equation that $B = 0$, so that equation (2) becomes $Ax^2 + Cy^2 + 2Ey = 0$, a conic symmetrical with respect to the Y-axis, which is consequently an axis of the curve. But the center of the circle lies on the Y-axis and hence on an axis of the curve, which was to be proved.

NOTE BY THE EDITORS.—The third equation gives $A = -E$, so that the equation of the conic becomes $Ax^2 + Cy^2 - 2Ay = 0$. But this apparent restriction is due to the choice of the radius r of the circle as unity. To return to the general case we have merely to make the transformation $X = rx$, $Y = ry$, which gives

$$AX^2 + CY^2 - 2E'Y = 0, \qquad r = E'/A,$$

where r is the radius of curvature at the extremity of an axis of the conic.

This problem may be handled without the use of derivatives by applying the theorem: If a conic and a circle intersect in four points, the angle between each of the three pairs of common chords is bisected by the perpendicular from their intersection to an axis of the conic. This may be proved by writing the equation of the conic without the xy term as $S = 0$, and the equation of the circle in the general form as $C = 0$. Then $S - kC = 0$ gives the equation of a pair of common chords by a suitable choice of k. Hence, for this value of k,

$$S - kC = l(y - mx + n)(y + mx + n'),$$

which proves the theorem.

A number of special theorems follow from this. We shall consider two which are related to this problem. By considering the limiting case in which the chords reduce to the tangent to the conic and the chord through the point of contact, we have the known theorem that the angle between the common tangent and the common chord of the conic and the osculating circle is bisected by the perpendicular from the point of osculation to an axis of the conic. If now we let the chord and tangent fall into coincidence, we have contact of the third order, and the tangent must now be perpendicular to an axis. But this can only happen at a vertex of the conic and where the center of the circle falls on an axis.

If, on the other hand, a pair of common chords coincide, this chord of double contact must be perpendicular to an axis, and here again the center of the circle lies on an axis.

Also solved by MAURICE BAUDIN, WILLIAM HOOVER, A. PELLETIER, A. V. RICHARDSON, J. K. WHITTEMORE and the PROPOSER.

2991 [1922, 356]. Proposed by E. J. OGLESBY, New York University.

Sum the infinite series

$$S_3(x) = 1 + \frac{3x^2}{2!} + \frac{4x^4}{4!} + \frac{6x^6}{6!} + \cdots,$$

where the numerators of the coefficients form a series of numbers whose third differences are all equal to 2.

Solution by Maurice Home, University of British Columbia.

The problem will be considered as a special case of the following summation:

$$S_p(x) = a_0 + a_p \frac{x^p}{p!} + a_{2p} \frac{x^{2p}}{(2p)!} + \cdots \qquad (p = \text{a positive integer}).$$

Let

$$f(x) = \sum_{r=0}^{r=\infty} \frac{a_r x^r}{r!}.$$

With the usual notation of finite differences,

$$f(x) = e^{xE} \cdot a_0 = e^{x(1+\Delta)} \cdot a_0 = e^x \cdot e^{x\Delta} \cdot a_0 = e^x \sum_{r=0}^{r=\infty} \Delta^r a_0 \frac{x^r}{r!}.$$

If now a_r is a rational integral function of r of degree n,

$$\Delta^j a_0 = 0 \qquad \text{for} \qquad j > n$$

and

$$f(x) = e^x \sum_{r=0}^{r=n} \Delta^r a_0 \frac{x^r}{r!}.$$

If p is prime, let θ be a special root of $x^p - 1 = 0$. Then it is readily found that

$$S_p(x) = \frac{1}{p} \sum_{j=0}^{j=p-1} f(\theta^j x).$$

The case of p not prime may be made to depend on the above solution. In the particular case given,

$$\cdot a_r = \frac{1}{24}(r^3 - 9r^2 + 38r + 24),$$

$$a_0 = 1, \quad \cdot \Delta a_0 = \frac{5}{4}, \quad \Delta^2 a_0 = -\frac{1}{2}, \quad \Delta^3 a_0 = \frac{1}{4},$$

$$\Delta^j a_0 = 0, \quad j > 3; \quad \theta = -1, \quad p = 2.$$

$$\begin{aligned} S_2(x) &= \frac{1}{2}\{f(x) + f(-x)\} \\ &= \frac{e^x}{2}\left(1 + \frac{5}{4}x - \frac{1}{4}x^2 + \frac{1}{24}x^3\right) + \frac{e^{-x}}{2}\left(1 - \frac{5}{4}x - \frac{1}{4}x^2 - \frac{1}{24}x^3\right) \\ &= \left(1 - \frac{x^2}{4}\right)\cosh x + \left(\frac{5x}{4} + \frac{x^3}{24}\right)\sinh x. \end{aligned}$$

Also solved by E. H. Clarke, A. Pelletier, J. F. Reilly, Elijah Swift, Pincus Schub, and the Proposer.

2992 [1922, 420]. Proposed by AUGUSTUS BOGARD, Teresian University, Winona, Wis.

A semicircle rotates at a uniform velocity about its diameter and slides along the line of that diameter at such a uniform rate as just to pass the full length of the diameter while making one revolution about it. Find the equation of the surface thus generated.

Solution by William Hoover, Columbus, Ohio.

Let the plane of the semi-circle in its initial position be the xz plane, with the diameter $OA = 2a$ along the z-axis. When the plane rotates through the angle θ toward the positive y-axis, the center C rises to the position (o, o, z_c) and we have from the conditions of the problem $(z_c - a)/a = \theta/\pi$. In the new position the equation of the semi-circle is $x' = \sqrt{a^2 - (z - z_c)^2}$ if we take as x'-axis the intersection of the xy-plane with the plane of the circle. Now if (x, y, z) are the coördinates of a point P on the semi-circle, we have at once

$$y = \sin\theta \sqrt{a^2 - \left(z - a - \frac{a\theta}{\pi}\right)^2},$$

$$x = \cos\theta \sqrt{a^2 - \left(z - a - \frac{a\theta}{\pi}\right)^2},$$

as the parametric equations of the surface generated by the *semi-circle*. The minus sign before the radical gives the surface generated by the other half of the circle. By eliminating θ we have for the equation of the surface generated by the entire circle

$$y = x \tan \frac{\pi}{a}(z - a \pm \sqrt{a^2 - x^2 - y^2}).$$

Also solved by MAURICE BAUDIN, THEODORE BENNETT, MALCOLM FOSTER, H. HALPERIN and A. PELLETIER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

A. D. PITCHER, Professor of Mathematics at Adelbert College, Western Reserve University, died on October 5, aged 43 years. While he had suffered from a weak heart for a number of years, his sudden death came as an unexpected blow to his family and colleagues. In fact he had met his classes as usual on October 4 and gave every indication of normal health.

Professor Pitcher was born and reared in Kansas and, having graduated from the University of Kansas in 1906 and received the degree of Doctor of Philosophy from the University of Chicago in 1910, he had held positions at the University of Kansas, at Dartmouth College, and during the past eight years at Adelbert College. Since receiving his doctorate, Professor Pitcher had carried on extensive researches in Moore's General Analysis, the Frechet Theory, and the Theory of Functions of a Real Variable, his contributions to these fields of mathematics giving him a high place in the esteem of American mathematicians. At all times he maintained a high standard of work, and because of his patient and sympathetic temperament he enjoyed an unusual degree of loyalty and affection from his students as well as the admiration of his associates.

Professor C. B. WILLIAMS, professor of mathematics and dean of Kalamazoo College, and his wife both lost their lives at Yokohama, Japan, in the recent severe earthquake in that country. They had left Kalamazoo during the summer expecting to spend the coming year in travel and study and with the intention of making a trip around the world.

Dr. J. N. VAN DER VRIES, charter member of the Association, has been appointed manager of the North Central Division of the Chamber of Commerce of the United States of America. Dr. Van der Vries was for seventeen years a member of the mathematical staff at the University of Kansas, and was chairman of the department from 1911 to 1917.

Professor ARCHIBALD HENDERSON, of the University of North Carolina, has a year's leave of absence on full pay from the Kenan Research Foundation. He sails for England on October 17 and will spend the year in research work on relativity in the Universities of Cambridge, Berlin, Rome and Paris.

Professor D. T. WILSON, of the department of astronomy of Case School of Applied Science, died on Friday, October 12, 1923, at Washington, D. C., after

a long illness. Dr. Wilson was born at Clinton, N. C., was graduated from the University of North Carolina in 1887, received his degree of A.M. from Vanderbilt University in 1896 and the degree of Ph.D. from the University of Chicago in 1905. He spent a number of years as a computer in the United States Observatory at Washington, then taught the two years 1901–03 at the University of Cincinnati, before coming to Case School in 1903 as an assistant professor. He was made associate professor in 1911. Illness compelled him to give up his work at the end of the college year in 1921. When the Warner and Swasey Observatory was being planned, Dr. Wilson assisted in the designing of the building and the equipment. He made a special study of ballistics and during the war taught classes in that subject, coöperating with the Naval War College and the Coast Artillery Corps. The special astronomical research to which he gave his attention was the computation of the perturbations of a group of asteroids, printed in Upsala in 1912.

Professor MALCOLM MCNEILL of Lake Forest University died suddenly on October 5, 1923. He was born at Galena, Illinois, 1855, graduated from Princeton University in 1877 where he became fellow in astronomy and received the Master's degree in 1880. He was instructor in astronomy at Princeton in 1881–2, then assistant professor until he came to Lake Forest in 1888 as professor and head of the department of astronomy. His work, however, was largely in the department of mathematics where he served the institution for 35 years.

Professor McNeill was greatly beloved by the students who, by a large majority vote, nominated him as the most popular professor in the institution and who were to have presented his portrait to the College as a feature of the homecoming day this fall. The portrait, which is a special gift of the class of 1923, will be hung in the Durrand Institute. Professor McNeill was a charter member of the Mathematical Association of America.

A letter just received by Professor D. E. SMITH from Mr. YOSHIO MIKAMI, the well-known Japanese writer upon the history of mathematics in his country, states that the Imperial Academy with which he is connected was not destroyed by the recent earthquake and fire. He also states that his own valuable collection of Japanese mathematical books and manuscripts is safe. For a time all this material was in danger of fire, but the wind changed, and it was thus saved. The University Library, however, and the Colleges of Letters, Justice, and Economics were entirely destroyed with a large quantity of valuable books, but most of the other important libraries were fortunately saved. It cannot fail to be a source of gratification to American scholars to know that at least the most valuable part of the historical mathematical libraries was saved.

On the two hundredth anniversary of the birth of Euler, a committee of the *Society of Swiss Naturalists* launched the project of international coöperation for the publication of his collected works. Academies, Societies, including the American Mathematical Society, and individuals subscribed for about 300 sets. Eighteen of the estimated seventy volumes have been published. By reason of the European War nearly one half of the subscribers have been unable to

meet their obligations in full. Under these circumstances, a considerable number of new subscribers must be secured if the completion of the undertaking is to be possible in the near future. Those libraries or individuals wishing information with a view to promoting this great international undertaking should communicate with Professor R. C. ARCHIBALD, Brown University, Providence, R. I.

Professor NIELS BOHR, of the University of Copenhagen, will deliver the Silliman lectures at Yale University, and also the Simpson lectures at Amherst College, during the present academic year.

Professor F. N. COLE, of Columbia University, has been granted leave of absence for the second half of the present academic year.

Dr. IRWIN ROMAN, of Northwestern University, has been appointed associate professor of mathematics at Vanderbilt University.

Mr. A. D. CAMPBELL, of Cornell University, has been appointed assistant professor of mathematics at the University of Arkansas.

Professor O. W. ALBERT, of Grinnell College, has been appointed head of the department of mathematics at the University of Redlands.

Mr. D. L. HOLL, of the University of Chicago, has been appointed assistant professor of mathematics at Ohio Wesleyan University.

Dr. G. M. ROBISON, of Cornell University, has been appointed assistant professor of mathematics at Trinity College, Durham.

Dr. G. E. RAYNOR, of Princeton University, has been appointed assistant professor of mathematics at Wesleyan University.

The following have been appointed to instructorships of mathematics: Dr. F. H. MURRAY, at Dalhousie University; Mr. T. ANDREW and Mr. J. N. NIXON, at the University of Kentucky; Mr. H. A. ROBINSON, at the Texas Agricultural and Mechanical College.

Professor C. N. LITTLE, dean of the college of engineering of the University of Idaho, died September 7, 1923. Professor Little was known for his contributions to the theory of knots.

Dr. H. W. WILSON, assistant professor of mathematics at the University of Iowa, has been granted a year's leave of absence on account of ill health; he is spending it at Albion, Michigan.

At the meeting in New York of the American Mathematical Society, officers were elected as follows: vice-president for one year, Professor E. V. HUNTINGTON, Harvard University; for two years, Professors T. H. HILDEBRANDT, University of Michigan and J. H. M. WEDDERBURN, Princeton University; secretary for two years, Professor R. G. D. RICHARDSON, Brown University; treasurer for two years, Professor W. B. FITE, Columbia University; librarian for three years, Professor R. C. ARCHIBALD, Brown University.

The Bôcher memorial prize for mathematical research was awarded to Professor G. D. BIRKHOFF, Harvard University, for his memoir on "Dynamical systems with two degrees of freedom."

The *Harvard Alumni Bulletin* for January, 1924, contains an extended article by Professor J. L. COOLIDGE entitled "The story of mathematics at Harvard."

Interest in the science at the university, it appears, may be dated from the year 1727 at which time the Hollis professorship in mathematics and natural philosophy was founded. The first incumbent was Isaac Greenwood, who was however afterwards dismissed for bad behavior. The famous Benjamin Peirce served from 1840 until 1880 but under the title of Perkins professor, the Hollis professorship having become ultimately identified with the department of physics. Aside from other details, the article contains photographic reproductions of paintings of Hollis and Peirce, also of the set of mathematical models to be seen in the Widener Library.

It is understood that the recent Act of Congress in reducing the army and navy will affect the instructional staff at the United States Naval Academy, several instructors in mathematics being automatically dropped at the end of this year. While the MONTHLY has never undertaken to conduct a bureau of information with respect to vacancies and candidates, it would seem that this situation at the Academy is sufficiently unique to warrant specific mention. Institutions which may be looking for instructors in mathematics may secure information concerning the men who will be losing their positions by addressing Professor H. E. Slaught, University of Chicago.

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